According to the right hand screw rule, if the screw is rotated in a direction from $\vec{P}$ to $\vec{Q}$ through the smaller angle, then the direction in which the tip of the screw advances is the direction of $\vec{R}$, perpendicular to the plane containing $\vec{P}$ and $\vec{Q}$. One example of vector or cross product is the force $\vec{F}$ experienced by a charge q moving with velocity $\vec{v}$ through a uniform magnetic field of magnetic induction $\vec{B}$. It is an empirical law (experimentally determined) given by $\vec{F} = q\vec{v} \times \vec{B}$.

Do you know ?

1. As linear displacement $\vec{x}$ is the distance travelled by a body along the line of travel, angular displacement $\vec{\theta}$ is the angle swept by a body about a given axis. The rate of change of angular displacement is the angular velocity denoted by $\vec{\omega}$. If a body is rotating about as axis, it possesses an angular velocity $\vec{\omega}$. If at a point at a distance $\vec{r}$ from the axis of rotation the body has linear velocity $\vec{v}$, then $\vec{v} = \vec{\omega} \times \vec{r}$.
2. An external force is needed to move a body from one point to other. Similarly to rotate a body about an axis passing through it, torque is required. Torque is a vector with its direction along the axis of rotation and magnitude describing the turning effect of force $\vec{F}$ acting on the body to rotate it about the given axis. Torque $\vec{\tau}$ is given as $\vec{\tau} = \vec{r} \times \vec{F}$, $\vec{r}$ being the perpendicular distance of a point on the body where the force is applied from the axis of rotation.

Characteristics of Vector Product:

(1) Vector product does not obey commutative law of multiplication.

$$\vec{P} \times \vec{Q} \neq \vec{Q} \times \vec{P} \qquad \text{--- (2.20)}$$

However, $|\vec{P} \times \vec{Q}| = |\vec{Q} \times \vec{P}|$ i.e., the magnitudes are the same but the directions are opposite to each other.

(2) The vector product obeys the distributive law of multiplication.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \qquad \text{--- (2.21)}$$

(3) Special cases of cross product

$$|\vec{P} \times \vec{Q}| = P\,Q \sin\theta \qquad \text{--- (2.22)}$$

(i) If $\theta = 0$. i.e., if the two nonzero vectors are parallel to each other, their vector product is a zero vector $|\vec{P} \times \vec{Q}| = P\,Q \cdot 0 = 0$

(ii) If $\theta = 180°$, i.e., if the two nonzero vectors are anti-parallel, their vector product is a zero vector $|\vec{P} \times \vec{Q}| = P\,Q \sin 180° = P\,Q \sin \pi = 0$

(iii) If $\theta = 90°$, i.e., if the two nonzero vectors are perpendicular to each other, the magnitude of their vector product is equal to the product of magnitudes of the two vectors.

$$|\vec{P} \times \vec{Q}| = P\,Q \sin 90° = P\,Q$$

Thus $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$

(4) If $\vec{P} = \vec{Q}$ then $|\vec{P} \times \vec{Q}| = |\vec{P} \times \vec{P}| = |\vec{Q} \times \vec{Q}| = 0$.

Thus $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(5) Let $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

and $\vec{Q} = Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}$

$$\vec{P} \times \vec{Q} = \left(P_x\hat{i} + P_y\hat{j} + P_z\hat{k}\right) \times \left(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}\right)$$

$$= P_xQ_x\left(\hat{i} \times \hat{i}\right) + P_xQ_y\left(\hat{i} \times \hat{j}\right) + P_xQ_z\left(\hat{i} \times \hat{k}\right)$$

$$+ P_yQ_x\left(\hat{j} \times \hat{i}\right) + P_yQ_y\left(\hat{j} \times \hat{j}\right) + P_yQ_z\left(\hat{j} \times \hat{k}\right)$$

$$+ P_zQ_x\left(\hat{k} \times \hat{i}\right) + P_zQ_y\left(\hat{k} \times \hat{j}\right) + P_zQ_z\left(\hat{k} \times \hat{k}\right)$$

Now $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$, and

$$\hat{i} \times \hat{k} = -\hat{j},\ \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}.$$

$$\therefore \vec{P} \times \vec{Q} = 0 + P_xQ_y\hat{k} - P_xQ_z\hat{j}$$

$$- P_yQ_x\hat{k} + 0 + P_yQ_z\hat{i}$$

$$+ P_zQ_x\hat{j} - P_zQ_y\hat{i} + 0$$

$$= (P_yQ_z - P_zQ_y)\,\hat{i}$$

$$+ (P_zQ_x - P_xQ_z)\,\hat{j}$$

$$+ (P_xQ_y - P_yQ_x)\,\hat{k}$$

This can be written in a determinant form as

$$\vec{P} \times \vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad \text{--- (2.23)}$$

(6) The magnitude of cross product of two vectors is numerically equal to the area of a parallelogram whose adjacent sides represent the two vectors.

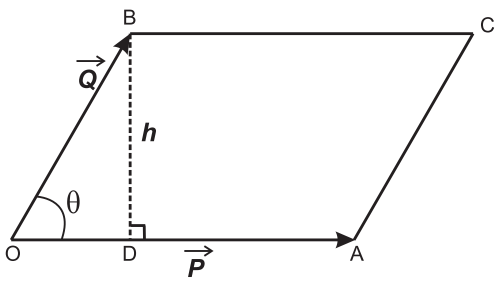


Fig 2.11: Area of parallelogram and vector product.

As shown in fig. 2.11,

$\vec{P} = \overrightarrow{OA}$, $\vec{Q} = \overrightarrow{OB}$, $\vec{P}$ and $\vec{Q}$ are inclined at an angle θ.

Perpendicular BD, of length h drawn on OA, gives the height of the parallelogram with OA as base.

Area of parallelogram

= base × height

$= \text{OA} \times \text{BD}$, as $\sin\theta = \dfrac{\text{BD}}{\text{OB}}$

$= P\,Q\sin\theta$

$= \left|\vec{P} \times \vec{Q}\right|$

= magnitude of the vector product --- (2.24)

Example 2.7: The angular momentum $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is a position vector and $\vec{p}$ is linear momentum of a body.

If $\vec{r} = 4\hat{i} \times 6\hat{j} - 3\hat{k}$ and $\vec{p} = 2\hat{i} + 4\hat{j} - 5\hat{k}$, find $\vec{L}$

Solution:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$\therefore \vec{L} = (-30 + 12)\hat{i} + (-6 + 20)\hat{j} + (16 - 12)\hat{k}$$

$$= -18\hat{i} + 14\hat{j} + 4\hat{k}.$$

Example 2.8: If $\vec{A} = 5\hat{i} + 6\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 3\hat{k}$, determine the angle between $\vec{A}$ and $\vec{B}$.

Solution: $\vec{A} \cdot \vec{B} = A\,B\cos\theta = A_xB_x + A_yB_y + A_zB_z$

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{A\,B}$$

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

$$\cos\theta = \frac{(5)(2) + (6)(-2) + (4)(3)}{\sqrt{25 + 36 + 16}\sqrt{4 + 4 + 9}}$$

$$= \frac{10}{\sqrt{77}.\sqrt{17}} = 0.2764$$

$$\theta = \cos^{-1} 0.2765 = 73°58'$$

Example 2.9: Given $\vec{P} = 4\hat{i} - \hat{j} + 8\hat{k}$ and $\vec{Q} = 2\hat{i} - m\hat{j} + 4\hat{k}$, find m if $\vec{P}$ and $\vec{Q}$ have the same direction.

Solution: Since $\vec{P}$ and $\vec{Q}$ have the same direction, their corresponding components must be in the same proportion, i.e.,

$$\frac{P_x}{Q_x} = \frac{P_y}{Q_y} = \frac{P_z}{Q_z}$$

$$\frac{4}{2} = \frac{-1}{-m} = \frac{8}{4}$$

$$\therefore m = \frac{1}{2}$$

2.6 Introduction to Calculus:

Calculus is the study of continuous (not discrete) changes in mathematical quantities. This branch of mathematics was first developed by G.W Leibnitz and Sir Issac Newton in the 17^{th} century and is extensively used in several branches of science. You will study calculus in mathematics in XII^{th} standard. Here we will learn the basics of the two branches of calculus namely differential and integral calculus. These are necessary to understand the topics covered in this book.

2.6.1 Differential Calculus:

Let us consider a function $y = f(x)$. Here x is called an independent variable and $f(x)$ gives the value of y for different values of x and is the dependent variable. For example x could be the position of a particle moving along x-axis and $y = f(x)$ could be its velocity at that position x. We can thus draw a graph of y against x as shown in Fig. 2.12 (a). Let A and B be two points on the curve giving values of y at $x = x_0$ and $x = x_0 + \Delta x$, where Δx is a small increment in x. The slope of the straight line joining A and B is given by $\tan\theta = \dfrac{\Delta y}{\Delta x}$.

If we make Δx smaller, the point B will come closer to A and if we keep making Δx smaller and smaller, we will ultimately reach a stage when B will coincide with A. This process is called taking the limit Δx going to zero and is written as $\lim\limits_{\Delta x \to 0}$. In this limit the line AB extended on both sides to P and Q will become the tangent to the curve at A, i.e., at

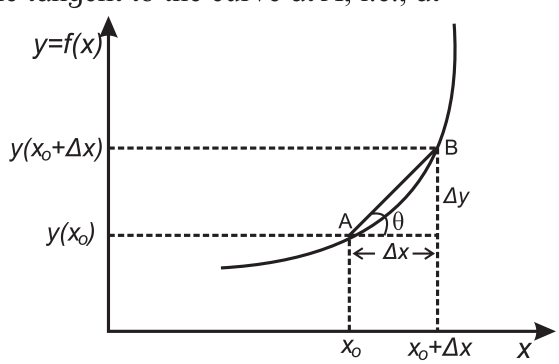


Fig. 2.12 (a): Average rate of change of y with respect to x.

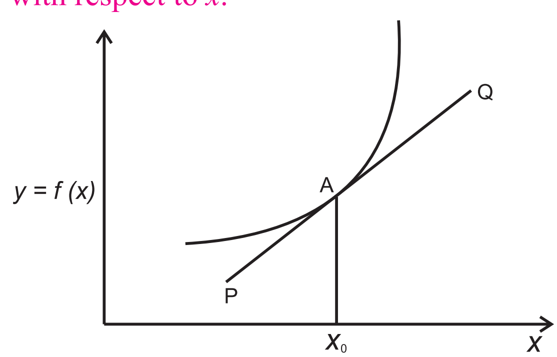


Fig. 2.12 (b): Rate of change of y with respect to x at x_0

$x = x_0$. In this limit both Δx and Δy will go to zero. However, when two quantities tend to zero, their ratio need not go to zero. In fact $\lim\limits_{\Delta x \to 0}\left(\dfrac{\Delta y}{\Delta x}\right)$ becomes the slope of the tangent shown by PQ in Fig. 2.12 (b). This is written as dy/dx at $x = x_0$.

Thus,

$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{\Delta x \to 0} \frac{(y + \Delta y) - y}{\Delta x}$$

$$\left.\frac{df(x)}{dx}\right|_{x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

We can drop the subscript zero and write a general formula which will be valid for all values of x as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{df(x)}{dx} \quad \text{--- (2.25)}$$

In XII[th] standard you will learn about the properties of derivatives and how to find derivatives of different functions. Here we will just list the properties as we will need them in later Chapter s. dy/dx is called the derivative of y with respect to x (which is the rate of change of y with respect to change in x) and the process of finding the derivative is called differentiation. Let $f_1(x)$ and $f_2(x)$ be two different functions of x and let s be a constant. Some of the properties of differentiation are

1. $$\frac{d(sf(x))}{dx} = s\frac{df(x)}{dx} \quad \text{--- (2.26)}$$

2. $$\frac{d}{dx}(f_1(x) + f_2(x)) = \frac{df_1(x)}{dx} + \frac{df_2(x)}{dx} \quad \text{--- (2.27)}$$

3. $$\frac{d}{dx}(f_1(x) \times f_2(x)) = f_1(x)\frac{df_2(x)}{dx} + f_2(x)\frac{df_1(x)}{dx} \quad \text{--- (2.28)}$$

4. $$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{1}{f_2(x)}\frac{df_1(x)}{dx} - \frac{f_1(x)}{f_2^2(x)}\frac{df_2(x)}{dx} \quad \text{--- (2.29)}$$

5. If x depends on time another variable t then,

$$\frac{df(x)}{dt} = \frac{df(x)}{dx}\frac{dx}{dt} \quad \text{--- (2.30)}$$

6.

$$\frac{d}{dx} f\left(g[x]\right) = f'\left(g(x)\right) \times g'(x)$$

where $f'\left(g(x)\right) = \dfrac{df}{dg}$

or $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx}$

The derivatives of some simple functions of x are given below.

1. $\dfrac{d}{dx}(x^n) = n\,x^{n-1}$ --- (2.31)

2. $\dfrac{d(e^x)}{dx} = e^x$ and $\dfrac{d(e^{ax})}{dx} = ae^{ax}$ --- (2.32)

3. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ --- (2.33)

4. $\dfrac{d}{dx}(\sin x) = \cos x$ --- (2.34)

5. $\dfrac{d}{dx}(\cos x) = -\sin x$ --- (2.35)

6. $\dfrac{d}{dx}(\tan x) = \sec^2 x$ --- (2.36)

7. $\dfrac{d}{dx}(\cot x) = -\mathrm{cosec}^2\, x$ --- (2.37)

8. $\dfrac{d}{dx}(\sec x) = \tan x \sec x$ --- (2.38)

9. $\dfrac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$ --- (2.39)

Example 2.10: Find the derivatives of the functions.

(a) $f(x) = x^8$ (b) $f(x) = x^3 + \sin x$

(c) $f(x) = x^3 \sin x$

Solution :

(a) Using $\dfrac{dx^n}{dx} = nx^{n-1}$,

$$\frac{d(x^8)}{dx} = 8x^7$$

(b) Using

$$\frac{d}{dx}(f_1(x) + f_2(x)) = \frac{df_1(x)}{dx} + \frac{df_2(x)}{dx} \quad \text{and}$$

$$\frac{d\,(\sin x)}{dx} = \cos x$$

$$\frac{d}{dx}(x^3 + \sin x) = \frac{d(x^3)}{dx} + \frac{d\,(\sin x)}{dx}$$

$$= 3x^2 + \cos x$$

c) Using

$$\frac{d}{dx}(f_1(x)\,f_2(x)) = f_1(x)\frac{df_2(x)}{dx} + \frac{df_1(x)}{dx}f_2(x)$$

and $\dfrac{d(\sin x)}{dx} = \cos x$

$$\frac{d}{dx}(x^3 \sin x) = x^3\frac{d\,(\sin x)}{dx} + \frac{d(x^3)}{dx}\sin x$$

$$= x^3 \cos x + 3x^2 \sin x$$

2.6.2 Integral calculus

Integral calculus is the branch of mathematics dealing with properties of integrals and their applications. Physical interpretation of integral of a function $f(x)$, i.e., $\int f(x)dx$ is the area under the curve $f(x)$ versus x. It is the reverse process of differentiation as we will see below.

We know how to find the area of a rectangle, triangle etc. In Fig. 2.13(a) we have shown y which is a function of x, A and B being two points on it.

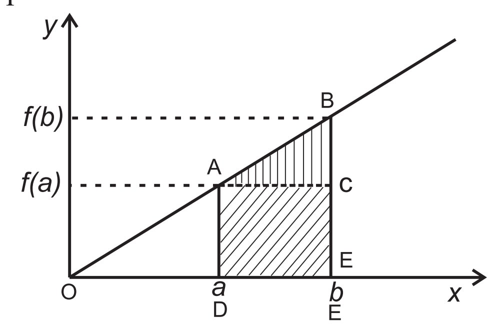


Fig. 2.13 (a): Area under a straight line.

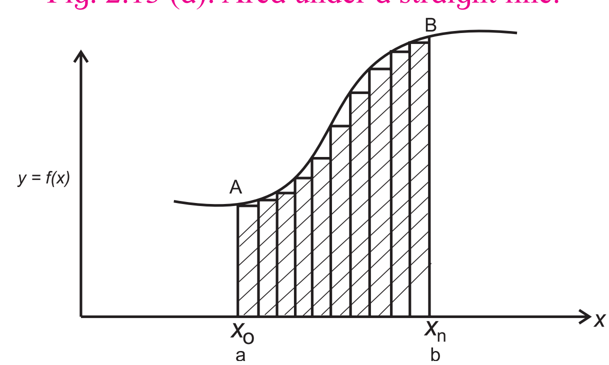


Fig. 2.13 (b): Area under a curve.

The area under the curve (straight line) from $x = a$ to $x = b$ is shown by shaded area. This can be obtained as sum of the area of the rectangle ADEC = $f(a)\ (b-a)$ and the area of the triangle ABC = 1/2 $(b-a)\ (f(b)-f(a))$

Figure 2.13(b) shows another function of x. We do not have a simple formula to calculate the area under this curve. For this calculation, we use a simple trick. We divide the area into a large number of vertical strips as shown in the figure. We assume thickness (width) of each strip to be so small that it can be assumed to be a rectangle as shown in the figure and add the areas of these rectangles. Thus the area under the curve is given by

Area under the curve

$$= \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i)$$

where n is the number of strips and ΔA_i is the area of the i^{th} strip.

As the strips are not really rectangles, the area calculated above is not exactly equal to the area under the curve. However as we increase n, the sum of areas of rectangles gets closer to the actual area under the curve and becomes equal to it in the limit $n \to \infty$. Thus we can write,

Area under the curve

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i) \qquad \text{--- (2.40)}$$

Integration helps us in getting exact area if the change is really continuous, i.e., n is really infinite. It is represented as $\int_{x=a}^{x=b} f(x)dx$ and is called the definite integral of $f(x)$ from $x = a$ to $x = b$.

Thus, $$\int_{x=a}^{x=b} f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i) \qquad \text{--- (2.41)}$$

The process of obtaining the integral is called integration. We can also write

$$F(x) = \int f(x)dx \qquad \text{--- (2.42)}$$

$F(x)$ is called the indefinite (without any limits on x) integral of $f(x)$. Differentiation is the reverse process to that of integration. Therefore,

$$f(x) = \frac{d}{dx}(F(x)) \qquad \text{--- (2.43)}$$

$$\therefore F(x)\Big|_a^b = F(b) - F(a) = \int_a^b f(x)dx \qquad \text{--- (2.44)}$$

Properties of integration

1. $\int \left(f_1(x) + f_2(x) \right) dx = \int f_1(x)dx + \int f_2(x)dx$ --- (2.45)

2. $\int K\, f(x)dx = K \int f(x)dx$ for K = constant --- (2.46)

Indefinite integrals of some basic functions are given below. Their definite integrals can be obtained by using the Eq. (2.44)

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1}$ --- (2.47)

2. $\int \dfrac{1}{x} dx = \ln x$ --- (2.48)

3. $\int \sin x\, dx = -\cos x$ --- (2.49)

4. $\int \cos x\, dx = \sin x$ --- (2.50)

5. $\int e^x\, dx = e^x$ --- (2.51)

Example 2.11: Evaluate the following integrals:

(a) $\int x^8 dx$

(b) $\int_2^5 x^2 dx$

(c) $\int (x + \sin x)\, dx$

Solution: (a) Using formula

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad \int x^8 dx = \frac{x^9}{9}$$

(b) Using Eq. (2.44),

$$\int_2^5 x^2 dx = \frac{x^3}{3}\Big|_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125-8}{3} = \frac{117}{3}$$

(c) Using Eq. (2.45),

$$\int \left(f_1(x) + f_2(x) \right) dx = \int f_1(x)dx + \int f_2(x)dx$$

and $\int \sin x\, dx = \cos x$, we get $\int (x + \sin x)\, dx$

$$\int x\, dx + \int \sin x\, dx = \frac{x^2}{2} - \cos x$$

Internet my friend

1. hyperphysics.phy-astr.gsu.edu/hbase/vect.html#veccon
2. hyperphysics.phy-astr.gsu.edu/hbase/hframe.html

Exercises

1. Choose the correct option.

i) The resultant of two forces 10 N and 15 N acting along $+x$ and $-x$-axes respectively, is

(A) 25 N along $+x$-axis
(B) 25 N along $-x$-axis
(C) 5 N along $+x$-axis
(D) 5 N along $-x$-axis

ii) For two vectors to be equal, they should have the

(A) same magnitude
(B) same direction
(C) same magnitude and direction
(D) same magnitude but opposite direction

iii) The magnitude of scalar product of two unit vectors perpendicular to each other is

(A) zero (B) 1
(C) -1 (D) 2

iv) The magnitude of vector product of two unit vectors making an angle of 60° with each other is

(A) 1 (B) 2
(C) 3/2 (D) $\sqrt{3}/2$

v) If $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors, then which of the following is not correct?

(A) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

(B) $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$

(C) $\vec{A}\times\vec{B}=\vec{B}\times\vec{A}$

(D) $\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{B}\times\vec{C}$

2. Answer the following questions.

i) Show that $\vec{a}=\dfrac{\hat{i}-\hat{j}}{\sqrt{2}}$ is a unit vector.

ii) If $\vec{v}_1=3\hat{i}+4\hat{j}+\hat{k}$ and $\vec{v}_2=\hat{i}-\hat{j}-\hat{k}$, determine the magnitude of $\vec{v}_1+\vec{v}_2$.

[Ans: 5]

iii) For $\vec{v}_1=2\hat{i}-3\hat{j}$ and $\vec{v}_2=-6\hat{i}+5\hat{j}$, determine the magnitude and direction of $\vec{v}_1+\vec{v}_2$.

$\left[\text{Ans: } 2\sqrt{5},\ \theta=\tan^{-1}\left(-\frac{1}{2}\right)\text{with } x\text{-axis}\right]$

iv) Find a vector which is parallel to $\vec{v}=\hat{i}-2\hat{j}$ and has a magnitude 10.

$\left[\text{Ans: }\frac{10}{\sqrt{5}}\hat{i}-\frac{20}{\sqrt{5}}\hat{j}\right]$

v) Show that vectors $\vec{a}=2\hat{i}+5\hat{j}-6\hat{k}$ and $\vec{b}=\hat{i}+\frac{5}{2}\hat{j}-3\hat{k}$ are parallel.

3. Solve the following problems.

i) Determine $\vec{a}\times\vec{b}$, given $\vec{a}=2\hat{i}+3\hat{j}$ and $\vec{b}=3\hat{i}+5\hat{j}$.

$\left[\text{Ans: }\hat{k}\right]$

ii) Show that vectors $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$, $\vec{b}=3\hat{i}-6\hat{j}+2\hat{k}$ and $\vec{c}=6\hat{i}+2\hat{j}-3\hat{k}$ are mutually perpendicular.

iii) Determine the vector product of $\vec{v}_1=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{v}_2=\hat{i}+2\hat{j}-3\hat{k}$,

$\left[\text{Ans: }-7\hat{i}+5\hat{j}+\hat{k}\right]$

iv) Given $\vec{v}_1=5\hat{i}+2\hat{j}$ and $\vec{v}_2=a\hat{i}-6\hat{j}$ are perpendicular to each other, determine the value of a.

$\left[\text{Ans: }\frac{12}{5}\right]$

v) Obtain derivatives of the following functions:

(i) $x\sin x$ (ii) $x^4+\cos x$
(iii) $x/\sin x$

$\left[\text{Ans: (i) }\sin x+x\cos x,\ \text{(ii) }4x^3-\sin x,\ \text{(iii) }\frac{1}{\sin x}-\frac{x\cos x}{\sin^2 x}\right]$

vi) Using the rule for differentiation for quotient of two functions, prove that

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\sec^2 x$$

vii) Evaluate the following integral:

(i) $\int_0^{\pi/2}\sin x\,dx$ (ii) $\int_1^5 x\,dx$

$\left[\text{Ans: (i) }1,\text{(ii) }12\right]$

***

3. Motion in a Plane

Can you recall?

1. What is meant by motion?
2. What is rectilinear motion?
3. What is the difference between displacement and distance travelled?
4. What is the difference between uniform and nonuniform motion?

3.1 Introduction:

We see objects moving all around us. Motion is a change in the position of an object with time. We have come across the motion of a toy car when pushed along some particular direction, the motion of a cricket ball hit by a batsman for a sixer and the motion of an aeroplane from one place to another. The motion of objects can be divided in three categories: (1) motion along a straight line, i.e., rectilinear motion, (2) motion in two dimensions, i.e., motion in a plane and, (3) motion in three dimensions, i.e., motion in space. The above cited examples correspond to three types of motions, respectively. You have studied rectilinear motion in earlier standards. In rectilinear motion the force acting on the object and the velocity of the object both are along one and the same line. The distances are measured along the line only and we can indicate distances along the +ve and –ve axes as being positive and negative, respectively. The study of the motion of an object in a plane or in space becomes much easier and the corresponding equations become more elegant if we use vector quantities. In this Chapter we will first recall basic facts about rectilinear motion. We will use vector notation for this study as it will be useful later when we will study the motion in two dimensions. We will then study the motion in two dimensions which will be restricted to projectile motion only. Circular motion, i.e., the motion of an object around a circular path will be introduced here and will be studied in detail in the next standard.

3.2 Rectilinear Motion:

Consider an object moving along a straight line. Let us assume this line to be along the x-axis. Let $\vec{x}_1$ and $\vec{x}_2$ be the position vectors of the body at times t_1 and t_2 during its motion. The following quantities can be defined for the motion.

1. **Displacement:** The displacement of the object between t_1 and t_2 is the difference between the position vectors of the object at the two instances. Thus, the displacement is given by

$$\vec{s} = \Delta\vec{x} = \vec{x}_2 - \vec{x}_1 \quad \text{--- (3.1)}$$

Its direction is along the line of motion of the object. Its dimensions are that of length. For example, if an object has travelled through 1 m from time t_1 to t_2 along the +ve x-direction, the magnitude of its displacement is 1 m and its direction is along the +ve x-axis. On the other hand, if the object travelled along the +ve y direction through the same distance in the same time, the magnitude of its displacement is the same as before, i.e., 1 m but the direction of the displacement is along the +ve y-axis.

2. **Path length**: This is the actual distance travelled by the object during its motion. It is a scalar quantity and its dimensions are also that of length. If an object travels along the x-axis from x = 2 m to x = 5 m then the distance travelled is 3 m. In this case the displacement is also 3 m and its direction is along the +ve x-axis. However, if the object now comes back to x = 4, then the distance through which the object has moved increases to 3 + 1 = 4 m. Its initial position was x = 2 m and the final position is now x = 4 m and thus, its displacement is Δx = 4 – 2 = 2 m, i.e., the magnitude of the displacement is 2 m and its direction is along the +ve x-axis. If the object now moves to x =1, then the distance travelled, i.e., the path length increases to 4 + 3 =

7 m while the magnitude of displacement becomes 2 – 1 = 1 m and its direction is along the negative x-axis.

3. **Average velocity**: This is defined as the displacement of the object during the time interval over which average velocity is being calculated, divided by that time interval. As displacement is a vector quantity, the velocity is also a vector quantity. Its dimensions are $[L^1 M^0 T^{-1}]$.

   If the position vectors of the object are $\vec{x}_1$ and $\vec{x}_2$ at times t_1 and t_2 respectively, then the average velocity is given by

   $$\vec{v}_{av} = \frac{\vec{x}_2 - \vec{x}_1}{(t_2 - t_1)} \quad \text{--- (3.2)}$$

   For example, if the positions of an object are $x = +2$ m and $x = +4$ m at times $t = 0$ and $t = 1$ minute respectively, the magnitude of its average velocity during that time is $v_{av} = (4 - 2)/(1 - 0) = 2$ m per minute and its direction will be along the +ve x-axis, and we write $\vec{v}_{av} = 2\hat{i}$ m/min where $\hat{i}$ is a unit vector along x-axis.

4. **Average speed**: This is defined as the total path length travelled during the time interval over which average speed is being calculated, divided by that time interval.

   Average speed = v_{av} = path length/time interval. It is a scalar quantity and has the same dimensions as that of velocity, i.e., $[L^1 M^0 T^{-1}]$.

   If the rectilinear motion of the object is only in one direction along a line, then the magnitude of its displacement will be equal to the distance travelled and so the magnitude of average velocity will be equal to the average speed. However if the object reverses its direction (the motion remaining along the same line) then the magnitude of displacement will be smaller than the path length and the average speed will be larger than the magnitude of average velocity.

5. **Instantaneous velocity**: Instantaneous velocity of an object is its velocity at a given instant of time. It is defined as the limiting value of the average velocity of the object over a small time interval (Δt) around t when the value of the time interval (Δt) goes to zero.

   $$\vec{v} = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{x}}{\Delta t}\right) = \frac{d\vec{x}}{dt}, \quad \text{--- (3.3)}$$

   $\frac{d\vec{x}}{dt}$ being the derivative of $\vec{x}$ with respect to t (see Chapter 2).

6. **Instantaneous speed:** Instantaneous speed is the speed of an object at a given instant of time t. It is the limiting value of the average speed of the object taken over a small time interval (Δt) around t when the time interval goes to zero. In such a limit, the path length will be equal to the magnitude of the displacement and so the instantaneous speed will always be equal to the magnitude of the instantaneous velocity of the object.

Always Remember:

For uniform rectilinear motion, i.e., for an object moving with constant velocity along a straight line

1. The average and instantaneous velocities are equal.
2. The average and instantaneous speeds are the same and are equal to the magnitude of the velocity.

For nonuniform rectilinear motion

1. The average and instantaneous velocities are different.
2. The average and instantaneous speeds are different.
3. The average speed will be different from the magnitude of average velocity.

Example 3.1: A person walks from point P to point Q along a straight road in 10 minutes, then turns back and returns to point R which is midway between P and Q after further 4 minutes. If PQ is 1 km, find the average speed and velocity of the person in going from P to R.

Solution: The path length travelled by the person is 1.5 km while the displacement is the distance between R and P which is 0.5 km. The time taken for the motion is 14 min.

The average speed = 1.5 / 14 = 0.107 km/min = 6.42 km/hr.

The magnitude of the average velocity = 0.5/14 = 0.0357 km/min = 2.142 km/hr.

Graphical Study of Motion

We can study the motion of an object by using graphs showing its position as a function of time. Figure 3.1 shows the graphs of position as a function of time for five different types of motion of an object. Figure 3.1(a) shows an object at rest, for which the *x-t* graph is a horizontal straight line. Since the position is not changing, displacement of the object zero. Velocity is displacement (which is zero) divided by time interval or the derivative of displacement with respect to time. It can be obtained from the slope of the line plotted in the figure which is zero.

Figure 3.1(b) shows *x-t* graph for an object moving with constant velocity along the +ve *x*-axis. Since velocity is constant, displacement is proportional to elapsed time. The slope of the straight line is +ve, showing that the velocity is along the +ve *x*-axis. As the motion is uniform, the average velocity is same as the instantaneous velocity at all times. Also, the speed is equal to the magnitude of the velocity.

Figure 3.1(c) shows the *x-t* graph for a body moving with uniform velocity but along the -ve *x*-axis, the slope of the line being -ve. Figure 3.1(d) shows the *x-t* graph of an object having oscillatory motion with constant speed. The direction of velocity changes from +ve to -ve and vice versa over fixed intervals of time.

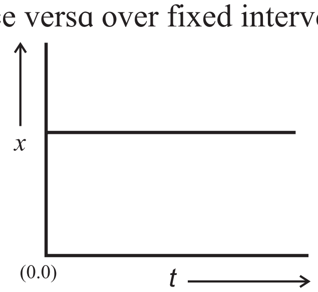


Fig 3.1 (a): Object at rest.

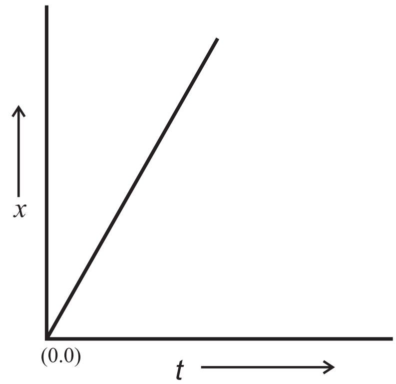


Fig 3.1 (b): Object with uniform velocity along +ve *x*-axis.

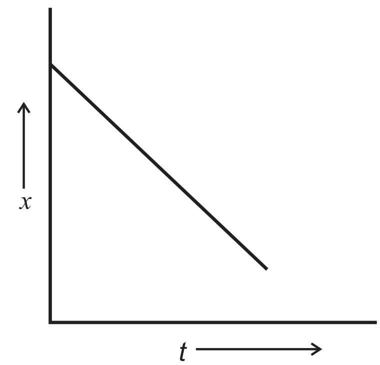


Fig 3.1 (c): Object with uniform velocity along -ve *x*-axis.

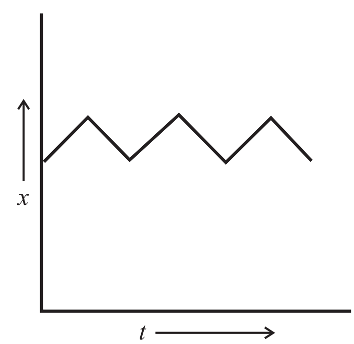


Fig 3.1 (d): Object performing oscillatory motion.

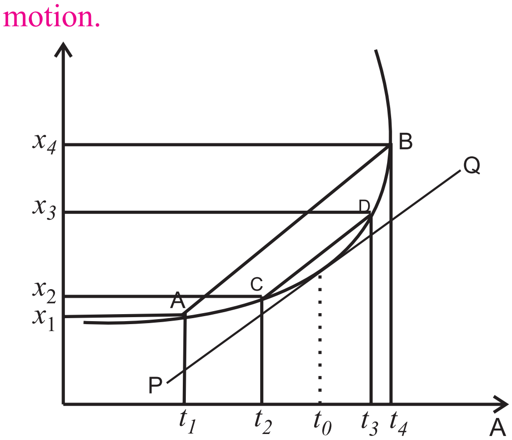


Fig.3.1 (e): Object in nonuniform motion.

Figure 3.1(e) shows the motion of an

object with nonuniform velocity. Its velocity changes with time and, therefore, the average and instantaneous velocities are different. Figure shows the average velocity over time interval from t_1 to t_4 around time t_0, which can be seen from Eq. (3.2) to be the slope of line AB. For a smaller time interval from t_2 to t_3, the average velocity is the slope of the line CD. If we keep reducing the time interval around t_0, we will ultimately come to a limit, when the time interval will go to zero and lines AB, CD... will go over to the tangent to the curve at t_0. The instantaneous velocity at t_0 will thus be equal to the slope of the tangent PQ at t_0 (see Eq. (3.3)).

7. Acceleration: Acceleration is defined as the rate of change of velocity with time. It is a vector quantity and its dimensions are $[L^1 M^0 T^{-2}]$. The average acceleration of an object having velocities $\vec{v}_1$ and $\vec{v}_2$ at times t_1 and t_2 is given by

$$\vec{a} = \frac{(\vec{v}_2 - \vec{v}_1)}{(t_2 - t_1)} \quad \text{--- (3.4)}$$

Instantaneous acceleration is the limiting value of the average acceleration when the time interval goes to zero. It is given by

$$\vec{a} = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{v}}{\Delta t}\right) = \frac{d\vec{v}}{dt} \quad \text{--- (3.5)}$$

The instantaneous acceleration at a given time is the slope of the tangent to the velocity versus time curve at that time. Figure 3.2 shows the velocity versus time (v - t) graphs for four different cases. Figure 3.2(a) represents the motion of an object with zero acceleration, i.e., constant velocity. The shaded area under the velocity-time graph over some time interval t_1 to t_2, shown in Figs. 3.2(a) is equal to $v_0 (t_2 - t_1)$ which is the magnitude of the displacement of the object from t_1 to t_2. Figure 3.2(b) is the velocity-time graph for an object moving with constant +ve acceleration (magnitude of velocity uniformly increasing with time). Figure 3.2(c) shows similar motion but the object has -ve acceleration, i.e., the acceleration is opposite to the direction of velocity which, therefore, decreases uniformly with time. The area under both the curves between two instants of time is the displacement of the object during that time interval (as shown below). Figure 3.2(d) shows the motion of an object having nonuniform acceleration. The average acceleration between t_1 and t_2 around t_0 and the instantaneous accelerations at t_0 for the object are shown by straight lines AB and CD respectively.

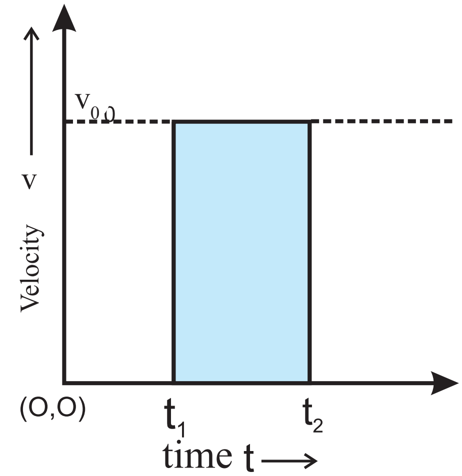


Fig 3.2 (a): Object moving with constant velocity.

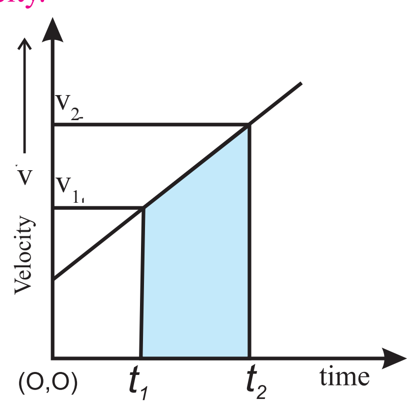


Fig 3.2 (b): Object moving with velocity (v) along +ve x-axis with uniform acceleration along the same direction.

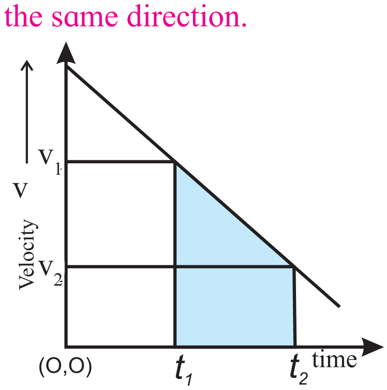


Fig 3.2 (c): Object moving with velocity (v) with negative uniform acceleration.

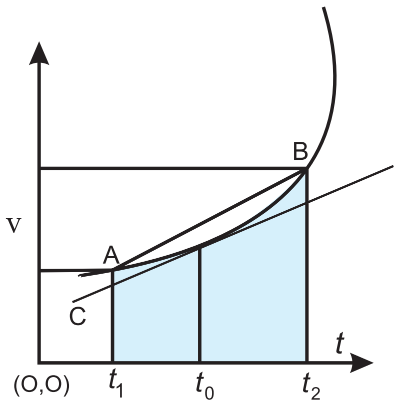


Fig. 3.2 (d): Object moving with nonuniform acceleration.

The area under the velocity-time curves in Figs. 3.2(a) to (d) can be written using the definition of integral given in Chapter 2 as

$$\text{Area} = \int_{t_1}^{t_2} \mathrm{v}\,dt = \int_{t_1}^{t_2} \frac{dx}{dt}dt = \int_{t_1}^{t_2} dx = x(t_2) - x(t_1) \quad \text{--- (3.6)}$$

= displacement of the object from t_1 to t_2.

Always Remember:

For uniform acceleration, for a rectilinear motion:

1. Velocity-time graph is linear.
2. The area under the velocity-time graph between two instants of time t_1 and t_2 gives the displacement of the object during that time interval.
3. The slope of the velocity-time graph is the acceleration of the object

For nonuniform acceleration in a rectilinear motion:

1. Velocity-time graph is nonlinear.
2. The area under the velocity-time graph between two instants of time t_1 and t_2 gives the displacement of the object during that time interval.
3. The instantaneous acceleration of the object at a given time is equal to the slope of the tangent to the curve at that point.

While using the concept of area under the curve, the origin of the velocity axis (for v-t graph) must be zero.

Equations of Motion for Uniform Acceleration:

We can graphically derive Newton's equations of motion for an object moving with uniform acceleration. Consider an object having position $x = 0$ at $t = 0$. Let the velocity at $t = 0$ be u and at time t be v. The graphical representation of motion is shown in Fig. 3.3. The acceleration is given by the slope of the line AB. Thus,

$$\text{Acceleration, } a = \frac{\mathrm{v} - u}{t - 0} = \frac{\mathrm{v} - u}{t}$$

$$\therefore \mathrm{v} = u + at \quad \text{--- (3.7)}$$

This is the first equation of motion.

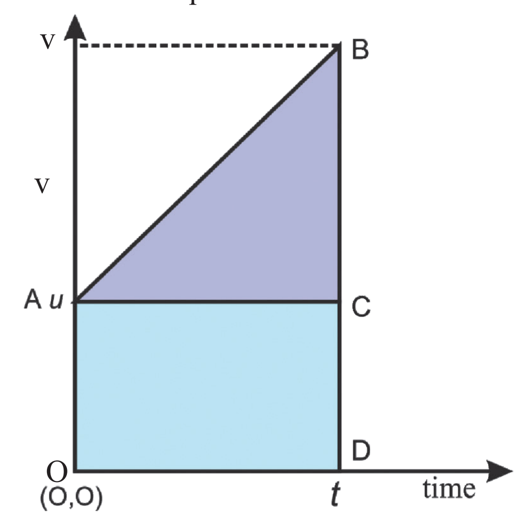


Fig.3.3: Derivation of equation of motion for motion with uniform acceleration.

As we know, the area under the curve in velocity-time graph is the displacement of the object. Thus displacement s = area of the quadrilateral OABD. = area of triangle ABC + area of rectangle OACD.

$$= \frac{1}{2}(v - u)t + ut$$

Using Eq. (3.7), $s = ut + \frac{1}{2}at^2$ --- (3.8)

This is the second equation of motion.

As the acceleration is constant, the velocity is increasing linearly with time and we can use average velocity v_{av}, to calculate the displacement using Eq. (3.7) as

$$s = \mathrm{v}_{av} t = \left(\frac{\mathrm{v} + u}{2}\right)t = \frac{(\mathrm{v} + u)(\mathrm{v} - u)}{2a}$$

$$\therefore s = \left(\mathrm{v}^2 - u^2\right)/(2a)$$

$$\therefore \mathrm{v}^2 - u^2 = 2a.s \quad \text{--- (3.9)}$$

This is the third equation of motion. Vector notation was not included here as the motion was rectilinear.

The most common example of uniform rectilinear motion with uniform acceleration of an object in day to day life is a freely falling body. When a body starts with zero velocity at a certain height from the ground and falls under the influence of the gravity of the Earth , it is said to be in free fall. The only other force that acts on it is that of the air resistance or friction. For displacements of a few metres, this force is too small and can be neglected. The acceleration of the body is the acceleration due to gravity which is along the vertical direction and can be assumed to be constant over distances which are small compared to the radius of the Earth . Thus the velocity and acceleration are both along the vertical direction and the motion is a uniform rectilinear motion with uniform acceleration.

Do you know ?

The distance travelled by an object starting from rest and having a uniform acceleration in successive seconds are in the ratio 1:3:5:7... Consider a freely falling object. Let us calculate the distances travelled by it in equal intervals of time t_0 (say). This can be done using the second equation of motion $s = u\,t_0 + (1/2)\,g\,t_0^2$. The initial velocity is zero. Therefore, the distance travelled in the first t_0 interval = $(1/2)\,g\,t_0^2$. For simplification let us write $(1/2)\,g$ = A. Then the distance travelled in the first t_0 time interval = $d_1 = At_0^2$. In the time interval $2t_0$, the distance travelled = $A(2t_0)^2$. Hence, the distance travelled in the second t_0 interval is $d_2 = A(4t_0^2 - t_0^2) = 3A\,t_0^2 = 3\,d_1$. The distance travelled in time interval $3t_0 = A(3t_0)^2$. Thus, the distance travelled in the 3rd t_0 interval = $d_3 = A(9t_0^2 - 4t_0^2) = 5A\,t_0^2 = 5d_1$. Continuing, one can see that the distances d_1, d_2, d_3 .. are in the ratio 1:3:5:7... This is true for any rectilinear motion, starting from rest, with positive uniform acceleration.

Example 3.2: A stone is thrown vertically upwards from the ground with a velocity 15 m/s. At the same instant a ball is dropped from a point directly above the stone from a height of 30 m. At what height from the ground will the stone and the ball meet and after how much time? (Use g = 10 m/s^2 for ease of calculation).

Solution: Let us assume that the stone and the ball meet after time t_0. The distances (not displacements) travelled by the stone and the ball in that time can be obtained from Eq. (3.8) as

$$s_{stone} = 15\,t_0 - \frac{1}{2}\,g\,t_0^2$$

$$s_{ball} = \frac{1}{2}\,g\,t_0^2$$

When they meet, $s_{stone} + s_{ball} = 30$

$$15\,t_0 - \frac{1}{2}\,g\,t_0^2 + \frac{1}{2}\,g\,t_0^2 = 30$$

$$t_0 = 30/15 = 2\text{ s}$$

$$\therefore\; s_{stone} = 15\,(2) - \frac{1}{2}\,(10)\,(2)^2 = 30 - 20 = 10\text{ m}$$

Thus the stone and the ball meet at a height of 10 m.

8. Relative Velocity: You must have often experienced relative motion. The most striking example is when you are going in a train and another train travelling in the same direction along parallel tracks, overtakes you. If you look at that train, it actually seems to be moving much slower than what your train seemed to move and yet it is overtaking you. On the other hand if your train overtakes another train, travelling on a parallel track in the same direction, and you look at that train, you feel that your train has suddenly slowed down. Why does this happen? This is because when you look at the neighbouring train, you are actually experiencing relative motion, i.e., your motion with respect to the other train or the motion of the other train with respect to you. Thus, in the first case as the other train overtakes you what you perceive is the velocity of the train with respect to you, i.e., the difference in the velocities of the two trains which most often is much smaller than the velocity of your train. In the second case, you are moving faster but when you look at that train you only feel your velocity relative to it and, therefore, your velocity appears to be lower than its actual value. We can define relative velocity of object A with respect to object B as the difference between their velocities, i.e.,

$$v_{AB} = v_A - v_B \qquad \text{--- (3.10)}$$

Similarly, the velocity of B with respect to A is given by

$$v_{BA} = v_B - v_A \qquad \text{--- (3.11)}$$

We assume that at time $t = 0$, A and B were at the same point $x = 0$. As they are travelling with different velocities, the distance between them will go on increasing with time in direct proportion to the difference in their velocities, i.e., the relative velocity between them.

Example 3.3: An aeroplane A, is travelling in a straight line with a velocity of 300 km/hr with respect to Earth. Another aeroplane B, is travelling in the opposite direction with a velocity of 350 km/hr with respect to Earth. What is the relative velocity of A with respect to B? What should be the velocity of a third aeroplane C moving parallel to A, relative to the Earth if it has a relative velocity of 100 km/hr with respect to A?

Solution: Let v_A, v_B and v_C be the velocities of the three planes relative to the Earth. Relative velocity of A with respect to B = $v_{AB} = v_A - v_B$ = 300 – (-350) = 650 km/hr

Relative velocity of C with respect to A = v_{CA} = $v_C - v_A$ = 100 km/hr.

Thus, $v_C = v_{CA} + v_A$ = 400 km/hr

3.3 Motion in Two Dimensions-Motion in a Plane:

So far we were considering rectilinear motion of an object. The direction of motion of the object was always along one straight line. Now we will consider the motion of an object in two dimensions, i.e., along a plane. Here, the direction of the force acting on an object will not be in the same line as its initial velocity. Thus, the velocity and acceleration will have different directions. For this reason we have to use vector equations. The definitions of various terms given in section 3.2 will remain valid except that the magnitude of the average velocity and the value of average speed will be different as the magnitude of the displacement need not be equal to the path length. For example, if a particle travels along a circle and comes back to its original position, its displacement will be zero but the path length will be equal to the circumference of the circle.

3.3.1 Average and Instantaneous Velocities:

For studying the motion of an object in two dimensions, for simplicity, we will take the plane to be the *x-y* plane. To describe the position of an object in this plane we will have to specify, both its *x* and *y* coordinates. The definitions of displacement, average and instantaneous velocities, average and instantaneous speeds and acceleration will be the same as those for rectilinear motion except that each of these quantities will now have components along the *x* and *y* directions. Let us assume the object to be at point P at time t_1 as shown in Fig. 3.4 (a).

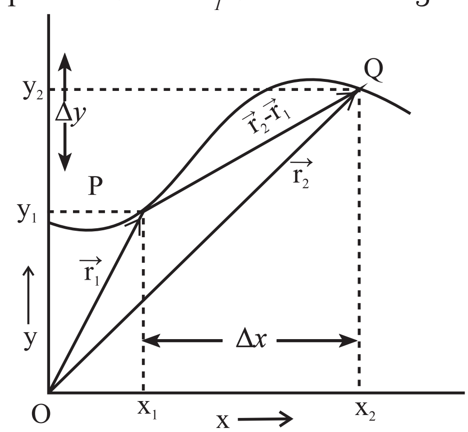


Fig. 3.4 (a) Motion in two dimensions

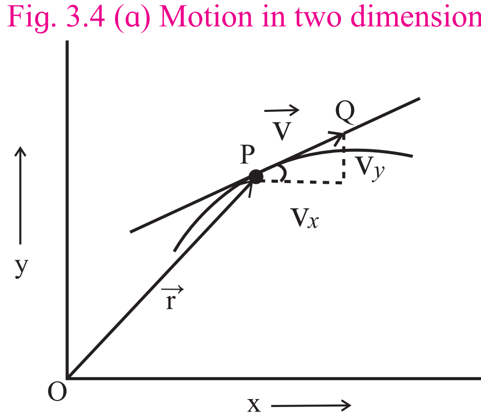


Fig. 3.4 (b) Instantaneous velocity

The position of the object will be described by its position vector $\vec{r}_1$. This can be written in terms of its components along the *x* and *y* axes as

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} \qquad \text{--- (3.12)}$$

At time t_2, let the position of the object be Q and its position vector be $\vec{r}_2$

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} \qquad \text{--- (3.13)}$$

The displacement of the particle from t_1 to t_2 shown by PQ, i.e., in time $t = t_2 - t_1$ is given by

$$\overrightarrow{\Delta r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} \qquad \text{--- (3.14)}$$

We can write the average velocity of the object as

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \left(\frac{x_2 - x_1}{t_2 - t_1}\right)\hat{i} + \left(\frac{y_2 - y_1}{t_{21} - t}\right)\hat{j}$$

$$\vec{v}_{av} = (v_{av})_x\hat{i} + (v_{av})_y\hat{j} \qquad \text{--- (3.15)}$$

where, $(v_{av})_x = (x_2 - x_1)/(t_2 - t_1)$ and

$$(v_{av})_y = (y_2 - y_1)/(t_2 - t_1) \qquad \text{--- (3.16)}$$

Average velocity is a vector whose direction is along $\Delta\vec{r}$ (see Eq. (3.2)), i.e., along the direction of displacement. In terms of its components, the magnitude (v) and direction (the angle θ that the velocity vector makes with the x-axis) can be written as (see Chapter 2)

$$v_{av} = \sqrt{(v_{av})_x^2 + (v_{av})_y^2} \quad \text{and}$$

$$\tan\theta = (v_{av})_y/(v_{av})_x \qquad \text{--- (3.17)}$$

Figure 3.4(b) shows the trajectory of an object moving in two dimensions. The instantaneous velocity of the object at point P along the trajectory is along the tangent to the curve at P. This is shown by the vector PQ. Its x and y components v_x and v_y are also shown in the figure.

The instantaneous velocity of the object can be written in terms of derivative as (see Eq. 3.3)

$$\vec{v} = \lim_{\Delta t\to 0}\left(\frac{\Delta\vec{r}}{\Delta t}\right) = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} \qquad \text{-- (3.18)}$$

The magnitude and direction of the instantaneous velocity are given by

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \qquad \text{--- (3.19)}$$

$$\tan\theta = (dy/dt)/(dx/dt) = dy \quad dx \qquad \text{--- (3.20)}$$

which is the slope of the tangent to the curve at the point at which we are calculating the instantaneous velocity.

3.3.2 Average and Instantaneous Acceleration:

Again, the definitions are the same as those for rectilinear motion. Thus, the average acceleration ($\vec{a}_{av}$) of a particle between times t_1 and t_2 can be written as

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \left(\frac{v_{2x} - v_{1x}}{t_2 - t_1}\right)\hat{i} + \left(\frac{v_{2y} - v_{1y}}{t_2 - t_1}\right)\hat{j} \qquad \text{- (3.21)}$$

where $\vec{v}_2$ and $\vec{v}_1$ are the velocities of the particle at times t_2 and t_1 respectively.

$$\vec{a}_{av} = (a_{av})_x\hat{i} + (a_{av})_y\hat{j} \qquad \text{--- (3.22),}$$

$(a_{av})_x$ and $(a_{av})_y$ being the x and y components of the average acceleration.

The magnitude and direction of the acceleration are given by

$$a_{av} = \sqrt{(a_{av})_x^2 + (a_{av})_y^2} \qquad \text{--- (3.23)}$$

and

$$\tan\theta = (a_{av})_y/(a_{av})_x \qquad \text{--- (3.24)}$$

The instantaneous acceleration is given by (see Eq. (3.5))

$$\vec{a} = \lim_{t\to 0}\left(\frac{\Delta\vec{v}}{\Delta t}\right) = \frac{d\vec{v}}{dt} = \left(\frac{dv_x}{dt}\right)\hat{i} + \left(\frac{dv_y}{dt}\right)\hat{j} \qquad \text{-(3.25)}$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)\hat{i} + \frac{d}{dt}\left(\frac{dy}{dt}\right)\hat{j} = \left(\frac{d^2x}{dt^2}\right)\hat{i} + \left(\frac{d^2y}{dt^2}\right)\hat{j} \qquad \text{--- (3.26)}$$

Thus, the x and y components of the instantaneous acceleration are respectively given by

$$a_x = d^2x/dt^2 \text{ and } a_y = d^2y/dt^2 \qquad \text{--- (3.27)}$$

The magnitude and direction of the instantaneous acceleration are given by

$$a = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} \qquad \text{--- (3.28),}$$

and

$$\tan\theta = (dv_y/dt)/(dv_x/dt) = dv_y/dv_x \qquad \text{--- (3.29)}$$

which is the slope of the tangent to the curve in velocity graph, i.e., a plot of v_y versus v_x.

Example 3.4: The position vectors of three particles are given by $\vec{x}_1 = (5\hat{i} + 5\hat{j})$ m, $\vec{x}_2 = (5t\hat{i} + 5t\hat{j})$ m and $\vec{x}_3 = (5t\hat{i} + 10t^2\hat{j})$m as a function of time t. Determine the velocity and acceleration for each, in SI units.

Solution: $\vec{v}_1 = d\vec{x}_1/dt = 0$ as $\vec{x}_1$ does not depend on time t.

Thus, the particle is at rest.

$\vec{v}_2 = d\vec{x}_2/dt = 5\hat{i} + 5\hat{j}$ m/s. $\vec{v}_2$ does not change with time. $\therefore \vec{a}_2 = 0$

$v_2 = \sqrt{5^2+5^2} = 5\sqrt{2}$ m/s, $\tan\theta = 5/5 = 1$ or $\theta = 45°$. Thus, the direction of v_2 makes an angle of 45° to the horizontal.

$\vec{v}_3 = d\vec{x}_3/dt = 5\hat{i} + 20t\hat{j}$.

$\therefore v_3 = \sqrt{5^2+(20t)^2}$ m/s. Its direction is along $\theta = \tan^{-1}\left(\frac{20t}{5}\right)$ with the horizontal.

$$\vec{a}_3 = \frac{d\vec{v}_3}{dt} = 20\hat{j} \text{ m/s}^2$$

Thus, the particle 3 is getting accelerated along the *y*-axis at 20 m/s^2.

3.3.3 Equations of Motion for an Object travellinging a Plane with Uniform Acceleration:

We have derived equations of motion for an object in rectilinear motion in section 3.2. We will now derive similar equations for a particle moving with uniform acceleration in two dimensions. Let the initial velocity of the object be $\vec{u}$ at $t = 0$ and its velocity at time t be $\vec{v}$. As the acceleration is constant, the average acceleration and the instantaneous acceleration will be equal. By using the definition of acceleration (Eq. (3.21)), we get

$$\vec{a} = (\vec{v} - \vec{u})/(t - 0)$$

$$\text{or } \vec{v} = \vec{u} + \vec{a}t \quad \text{--- (3.30)}$$

which is the same as Eq. (3.7) but is in vector form.

Let the displacement from time $t = 0$ to t be $\vec{s}$. This can be calculated from the average velocity of the object during this time. For constant acceleration, $\vec{v}_{av} = \frac{\vec{u}+\vec{v}}{2}$

$$\therefore \vec{s} = (\vec{v}_{av})t = \left(\frac{\vec{u}+\vec{v}}{2}\right)t = \left(\frac{\vec{u}+\vec{u}+\vec{a}t}{2}\right)t$$

$$\therefore \vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2 \quad \text{--- (3.31),}$$

which is the vector form of Eq. (3.8).

Eq. (3.30) and (3.31) can be resolved into their x and y components so as to get corresponding scalar equations as follows.

$$v_x = u_x + a_x t \quad \text{--- (3.32)}$$

$$\text{and} \quad v_y = u_y + a_y t \quad \text{--- (3.33)}$$

$$s_x = u_x t + \frac{1}{2}a_x t^2 \quad \text{--- (3.34)}$$

$$\text{and} \quad s_y = u_y t + \frac{1}{2}a_y t^2 \quad \text{--- (3.35)}$$

We can see that Eqs. (3.32) and (3.34) involve only the x components of displacement, velocity and acceleration while Eqs. (3.33) and (3.35) involve only the y components of these quantities. Thus the two sets of equations are independent of each other and can be solved independently. We can thus see that the motion along the x direction of an object is completely controlled by the x components of velocity and acceleration while that along the y direction is completely controlled by the y components of these quantities. This makes it easy to study the motion in two dimensions which gets converted to two independent rectilinear motions along two perpendicular directions.

Always Remember:

Motion in two dimensions can be resolved into two independent motions in mutually perpendicular directions.

Example 3.5: The initial velocity of an object is $\vec{u} = 5\hat{i} + 10\hat{j}$ m/s. Its constant acceleration is $\vec{a} = 2\hat{i} + 3\hat{j}$ m/s^2. Determine the velocity and the displacement after 5 s.

Solution:

$$\vec{v} = \vec{u} + \vec{a}t$$

$$= (5\hat{i} + 10\hat{j}) + (2\hat{i} + 3\hat{j})(5) = 15\hat{i} + 25\hat{j}$$

$$\therefore v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{15^2+25^2} = \sqrt{225+625} = \sqrt{850}$$

$$= 29.15 \text{ m/s}$$

Direction of $\vec{v}$ with *x*-axis is $\tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}$

$\left(\frac{25}{15}\right) = \tan^{-1}(1.667) = 59^{o}$

$$\vec{s} = \vec{u}\,t + \frac{1}{2}\vec{a}\,t^2$$

$$= \left(5\hat{i} + 10\hat{j}\right)(5) + \frac{1}{2}\left(2\hat{i} + 3\hat{j}\right)5^2$$

$$= 50\hat{i} + (87.5)\,\hat{j}$$

$$\therefore s = \sqrt{s_x^2 + s_y^2} = \sqrt{50^2 + 87.5^2}$$

$$= \sqrt{2500 + 7656.25}$$

$$= \sqrt{10156.25} = 100.78\ \text{m}$$

at $\tan^{-1}\frac{87.5}{50} = 60°15'$ with x-axis.

3.3.4 Relative Velocity:

Relative velocity between two objects moving in a plane can be defined in a way similar to that for objects moving along a straight line. The relative velocity of object A having velocity $\vec{v}_A$, with respect to the object B having velocity $\vec{v}_B$, is given by

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B \qquad \text{--- (3.36)}$$

Similarly, the relative velocity of object B with respect to object A , is given by

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A \qquad \text{--- (3.37)}$$

We can see that the magnitudes of the two relative velocities (v_{AB} and v_{BA}) are equal and their directions are opposite.

Consider a number of objects A, B, C, D ---- Y, Z, moving with respect to the other. Using the symbol v_{AB} for representing the velocity of A relative to B etc, the velocity of A relative to Z can be written as

$$\vec{v}_{AZ} = \vec{v}_{AB} + \vec{v}_{BC} + \vec{v}_{CD} + \ldots + \vec{v}_{XY} + \vec{v}_{YZ}$$

Note the order of subscripts (A→B→C→D---→Z).

Example 3.6: An aeroplane is travelling northward with a speed of 300 km/hr with respect to the Earth, when the wind is blowing from east to west at a speed of 100 km/hr. What is the velocity of the aeroplane with respect to the wind?

Solution: Let the velocity of the aeroplane with respect to Earth be $\vec{v}_{AE}$, velocity of wind with respect to Earth be $\vec{v}_{WE}$. The velocity of aeroplane with respect to wind, $\vec{v}_{AW}$ can be determined by the following expression:

$\vec{v}_{AW} = \vec{v}_{AE} + \vec{v}_{EW} = \vec{v}_{AE} - \vec{v}_{WE} = 100\hat{i} + 300\hat{j}$, considering north along +y axis.

Magnitude of $\vec{v}_{AW} = \sqrt{(10000 + 90000)}$

$= 100\sqrt{10}$ km/hr, and its direction,

$\theta = \tan^{-1}\left(\frac{300}{100}\right) = 71.6°$ is towards north of east.

3.3.5 Projectile Motion:

Any object in flight after being thrown with some velocity is called a projectile and its motion is called projectile motion. We often see projectile motion in our day-to-day life. Children throw stones towards trees for getting tamarind pods or mangoes. A bowler bowls a ball towards a batsman in cricket, a basket ball player throws a ball towards the basket, all these are illustrations of projectile motion. In this motion, we have objects (projectiles) with given initial velocity, moving under the influence of the Earth's gravitational field. The projectile has two components of velocity, one in the horizontal, i.e., along x-direction and the other in the vertical, i.e., along the y direction. The acceleration due to gravity acts only along the vertically downward direction. The horizontal component of velocity, therefore, remains unchanged as no force is acting in the horizontal direction, while the vertical component changes in accordance with laws of motion with $\vec{a}_x$ being 0 and $\vec{a}_y$ $(= -\vec{g}\,)$ being the downward acceleration due to gravity (upward is positive). Unless stated otherwise, retarding forces like air resistance, etc., are neglected for the projectile motion.

Let us assume that the initial velocity of the projectile is $\vec{u}$ and its direction makes an angle θ with the horizontal as shown in Fig. 3.5. The projectile is thrown from the ground. We take the x-axis along the ground and y-axis in the vertical direction. The horizontal and vertical components of initial velocity are u

cosθ and u sinθ respectively. The horizontal component remains unchanged in absence of any force acting in that direction, while the vertical component changes according to (Eq. 3.33) with $a_y = -g$ and $u_y = u \sin\theta$.

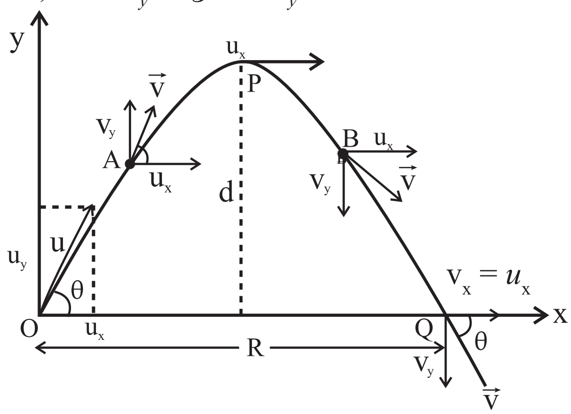


Fig.3.5: Trajectory of a projectile.

Thus, the components of velocity at time t are given by

$$v_x = u_x = u\cos\theta \quad \text{--- (3.38)}$$

$$v_y = u_y - gt = u\sin\theta - gt \quad \text{--- (3.39)}$$

As $0 < \theta < 90°$, the vertical component initially is in the upward direction. Similarly, the displacements of the projectile in the horizontal and vertical directions at time t, according to Eqs. (3.34) and (3.35) are given by

$$s_x = u\cos\theta . t \quad \text{---- (3.40)}$$

$$s_y = u\sin\theta . t - \frac{1}{2}gt^2 \quad \text{--- (3.41)}$$

The direction of motion of the projectile at any time t makes an angle α with the horizontal which is given by

$$\tan\alpha = v_y(t)/v_x(t) \quad \text{--- (3.42)}$$

The vertical velocity keeps on decreasing as the projectile goes up and becomes zero at certain time. At that time the height of the projectile is maximum. The velocity then starts increasing in the downward direction as the particle is now falling under the Earth 's gravitational field with a constant horizontal component of velocity. After a while the projectile reaches the ground. The trajectory of the object is shown in Fig. 3.5. The projectile is assumed to start from the origin of the coordinate system, O. The point of maximum height is indicated by P and the point where it falls down to the ground is indicated by Q. The horizontal and vertical components of velocity are shown at these points as well as at two intermediate points A and B, on the trajectory of the projectile. Note that the horizontal component of velocity remains the same, i.e., u_x, while the vertical component decreases and becomes zero at P. After that it changes its direction, its magnitude increases and becomes equal to u_y again at Q. The horizontal distance covered by the projectile before it falls to the ground is OQ. We can derive the equation of the trajectory of the projectile as follows.

Let the time taken by the projectile to reach the maximum height be t_0. The trajectory of the object being symmetrical, it can be shown by using equations of motion, that the object will take the same time in going up in air and coming down to the ground. At the highest point P, $t = t_0$ and $v_y = 0$. Using Eq. (3.39),

we get, $0 = u\sin\theta - gt_0$

$$t_0 = (u\sin\theta)/g \quad \text{--- (3.43)}$$

∴ Total time in air = $T = 2t_0$ is the ***time of flight***.

The total horizontal distance travelled by the particle in this time T can be obtained by using Eq. (3.40) as

$$R = u_x . T = u\cos\theta . 2t_0 = u\cos\theta . (2u\sin\theta)/g$$

$$= 2\, u_x\, u_y/g = u^2(2\sin\theta\cos\theta)/g$$

$$= u^2\sin 2\theta/g \quad \text{--- (3.44)}$$

This maximum horizontal distance travelled by the projectile is called the ***horizontal range*** R of the projectile and depends on the magnitude and direction of initial velocity of the projectile as well as the value of acceleration due to gravity at that place.

For maximum horizontal range,

$\sin 2\theta = 1 \;\therefore 2\theta = 90^0 \;or\; \theta = 45^0$

Hence, $R = R_{max} = \dfrac{u^2}{g}$ for $\theta = 45^0$

The ***maximum height*** H reached by the projectile, having certain value of θ, is the distance travelled along the vertical (y) direction in time t_0. This can be calculated by using Eq. (3.41) as

$$H = u\sin\theta . t_0 - \frac{1}{2}g t_0^2$$

$$= u\sin\theta\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$= \frac{u^2\sin^2\theta}{2g} = \frac{u_y^2}{2g} \qquad \text{--- (3.45)}$$

Do you know ?

All the above expressions of T, R, R_{max} and H are valid if the entire motion is governed only by gravitational acceleration g, i.e., retarding forces like air resistance are absent. However, in reality, it is never so. As a result, time of ascent t_a and time of decent t_d are not equal but $t_a > t_d$. Also, in order to achieve maximum horizontal range for given initial velocity, the angle of projection should be greater than 45^0 and the range is much less than $\frac{u^2}{g}$.

Example 3.7: A stone is thrown with an initial velocity components of 20 m/s along the vertical, and 15 m/s along the horizontal direction. Determine the position and velocity of the stone after 3 s. Determine the maximum height that it will reach and the total distance travelled along the horizontal on reaching the ground. (Assume g = 10 m/s^2)

Solution: The initial velocity of the stone in x-direction = $u\cos\theta$ = 15 m/s and in y-direction = $u\sin\theta$ = 20 m/s.

After 3 s, $v_x = u\cos\theta$ = 15 m/s and $v_y = u\sin\theta - gt$ = 20 – 10(3)= - 10 m/s = 10 m/s downwards.

$$\therefore v = \sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 10^2}$$

$$= \sqrt{225+100} = \sqrt{325}$$

$$= 18.03 \text{ m/s}$$

$\tan\alpha = v_y / v_x$ = 10/15 = 2/3

$\therefore \alpha = \tan^{-1}(2/3)$ = 33° 41' with the horizontal.

$s_x = (u\cos\theta)\,t$ = 15×3 = 45 m,

$s_y = (u\sin\theta)\,t - \frac{1}{2}gt^2$ = 20 × 3 - 5(3)^2 = 15 m.

Thus the stone will be at a distance 45 m along horizontal and 15 m along vertical direction from the initial position after time 3 s. The velocity is 18.03 m/s making an angle 33° 41' with the horizontal.

The maximum vertical distance travelled is given by $H = (u\sin\theta)^2/(2g) = 20^2/(2\times10)$ = 20 m

Maximum horizontal distance travelled

R = $2.u_x.u_y/g$ = 2(15)(20)/10 = 60 m

Equation of motion for a projectile

We can derive the equation of motion of the projectile which is the relation between the displacements of the projectile along the vertical and horizontal directions. This can be obtained by eliminating t between the equations giving these displacements, i.e., Eqs. (3.40) and (3.41).

As the projectile starts from $\vec{x} = 0$, we can write $s_x = x$ and $s_y = y$.

$$\therefore s_x = (u\cos\theta)t \quad \therefore t = \frac{s_x}{u\cos\theta} = \frac{x}{u\cos\theta}$$

$$\therefore y = (u\sin\theta)t - \frac{1}{2}gt^2$$

$$= (u\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\therefore y = (\tan\theta)x - \frac{1}{2}\left(\frac{g}{u^2\cos^2\theta}\right)x^2 \qquad \text{--- (3.46)}$$

This is the equation of the trajectory of the projectile. Here, u and θ are constants for the given projectile motion. The above equation is of the form

$$y = Ax + Bx^2 \qquad \text{--- (3.47)}$$

which is the equation of a parabola. Thus, the path, i.e., the trajectory of a projectile is a parabola.

3.4 Uniform Circular Motion:

An object moving with *constant speed* along a circular path is said to be in *uniform* circular motion (UCM). Such a motion is only possible if its velocity is *always tangential* to its circular path, *without change in its magnitude*.

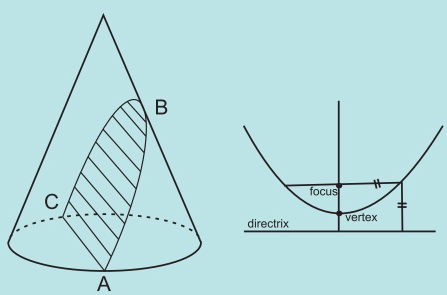
To change the direction of velocity, acceleration is a must. However, if the acceleration or its component is in line with the velocity (along or opposite to the velocity), it will *always* change the speed (magnitude of velocity) in which case it will not continue its uniform circular motion. In order to achieve both these requirements, the acceleration must be (i) perpendicular to the tangential velocity, (ii) of constant magnitude and (iii) always directed towards the centre of the circular trajectory. Such an acceleration is called *centripetal* (centre seeking) acceleration and the force causing this acceleration is *centripetal* force.

Thus, in order to realize a circular motion, there are two requirements; (i) *tangential velocity* and (ii) *centripetal force* of *suitable constant magnitude*.

An example is the motion of the moon going around the Earth in an early circular orbit as a result of the constant gravitational attraction of fixed magnitude felt by it towards the Earth.

Do you know ?

A parabola is a symmetrical open curve obtained by the intersection of a cone with a plane which is parallel to its side. Mathematically, the parabola is described with the help of a point called the focus and a straight line called the directrix shown in the accompanying figure. The parabola is the locus of all points which are equidistant from the focus and the directrix. The chord of the parabola which is parallel to the directrix and passes through the focus is called latus rectum of the parabola as shown in the accompanying figure.



3.4.1 Period, Radius Vector and Angular Speed:

Consider an object of mass m, moving with a uniform speed v, along a circle of radius r. Let T be the time period of revolution of the object, i.e., the time taken by the object to complete one revolution or to travel a distance of $2\pi r$.

Thus, $T = 2\pi r/\text{v}$

$$\therefore \text{Speed v} = \frac{\text{Distance}}{\text{Time}} = \frac{2\pi r}{T} \quad \text{--- (3.48).}$$

During circular motion of a point object, the position vector of the object from centre of the circle is the radius vector $\vec{r}$. Its magnitude is radius r and it is directed away from the centre to the particle, i.e., away from the centre of the circle. As the particle performs UCM, this radius vector describes equal angles in equal intervals of time. At this stage we can define a new quantity called angular speed ω which gives the angle described by the radius vector, per unit time. It is analogous to speed which is distance travelled per unit time.

During one complete revolution, the angle described is 2π and the time taken is period T. Hence, the angular speed

$$\omega = \frac{\text{Angle}}{\text{time}} = \frac{2\pi}{T} = \frac{(2\pi)}{\left(\frac{2\pi r}{\text{v}}\right)} = \frac{\text{v}}{r} \quad \text{--- (3.49)}$$

The unit of ω is radian/sec.

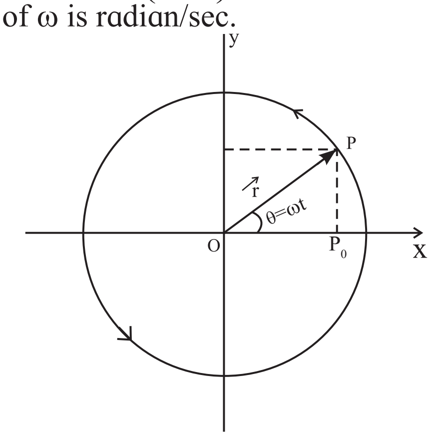


Fig.3.6: Uniform circular motion.

3.4.2 Expression for Centripetal Acceleration:

Figure 3.6 shows a particle P performing a UCM in anticlockwise sense along a circle of radius r with angular speed ω and period T. Let us choose the coordinates such that this motion is in the xy- plane having centre at the origin O. Initially (for simplicity), let the particle be at P_0 on the positive x-axis. At a given instant t, the radius vector of P makes an angle θ with the x-axis.

$$\therefore \theta = \omega t \text{ and so } \frac{d\theta}{dt} = \omega$$

x and y components of the radius vector $\vec{r}$ will then be $r\cos\theta$ and $r\sin\theta$ respectively.

$$\therefore \vec{r} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$= (r\cos[\omega t])\hat{i} + (r\sin[\omega t])\hat{j} \quad \text{--- (3.50)}$$

Time derivative of position vector $\vec{r}$ gives instantaneous velocity $\vec{v}$ and time derivative of velocity $\vec{v}$ gives instantaneous acceleration $\vec{a}$. Magnitudes of r and ω are constants.

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} = r\left(-\omega \sin[\omega t]\hat{i} + \omega\cos[\omega t]\hat{j}\right)$$

$$= r\omega\left(-\sin[\omega t]\hat{i} + \cos[\omega t]\hat{j}\right) \quad \text{--- (3.51)}$$

$$\therefore \vec{a} = \frac{d\vec{v}}{dt} = r\omega\left(-\omega\cos[\omega t]\hat{i} - \omega\sin[\omega t]\hat{j}\right)$$

$$= -\omega^2\left(r\cos[\omega t]\hat{i} + r\sin[\omega t]\hat{j}\right) = -\omega^2\vec{r} \quad \text{--- (3.52)}$$

Here minus sign shows that the acceleration is opposite to that of $\vec{r}$, i.e., towards the centre. This is the centripetal acceleration.

The magnitude of acceleration,

$$a = \omega^2 r = \frac{v^2}{r} = \omega v \quad \text{--- (3.53)}$$

The force providing this acceleration should also be along the same direction, hence centripetal.

$$\therefore \vec{F} = m\vec{a} = -m\omega^2\vec{r} \quad \text{--- (3.54)}$$

Magnitude of $F = m\omega^2 r = \frac{mv^2}{r} = m\omega v$ - (3.55)

Conical pendulum

In a simple pendulum a mass m is suspended by a string of length l and moves along an arc of a vertical circle. If the mass instead revolves in a horizontal circle and the string which makes a constant angle with the vertical describes a cone whose vertex is the fixed point O, then mass-string system is called a conical pendulum as shown in Fig. 3.7. In the absence of friction, the system will continue indefinitely once started.

As shown in the figure, the forces acting on the bob of mass, m, of the conical pendulum are: (i) Gravitational force, mg, acting vertically downwards, (ii) Force due to tension $\vec{T}$ acting along the string directed towards the support. These are the only two forces acting on the bob.

For the bob to undergo horizontal circular motion, (on a circle of radius r) the resultant force must be centripetal, (directed towards the centre of the circle). In other words vertical gravitational force must be balanced.

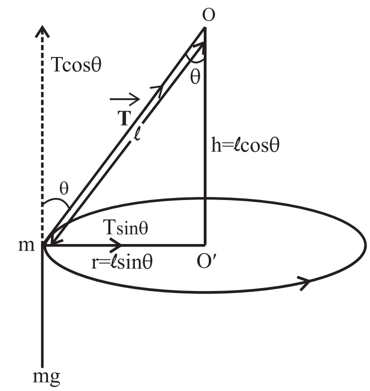


Fig 3.7: Conical pendulum

Thus, we resolve tension $\vec{T}$ into two mutually perpendicular components. Let θ be the angle made by the string with the vertical at any position. The component $T\cos\theta$ is acting vertically upwards. The inclination should be such that $T\cos\theta = mg$, so that there is no net vertical force.

The resultant force on the bob is then $T\sin\theta$ which is radial or centripetal or directed towards centre O'.

$$\therefore \quad T\sin\theta = mv^2/r$$

$$\tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{(mv^2/r)}{mg} = \frac{v^2}{rg}$$

Since we know $v = \frac{2\pi r}{T}$

$$\therefore \tan\theta = \frac{4\pi^2 r^2}{T^2 rg}$$

$$T = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

$$T = 2\pi\sqrt{\frac{l\sin\theta}{g\tan\theta}} \quad (\because r = l\sin\theta)$$

$$T = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

$$T = 2\pi\sqrt{\frac{h}{g}} \quad (\because h = l\cos\theta) \quad \text{--- (3.56)}$$

where l is length of the pendulum and h is the vertical distance of the horizontal circle from the fixed point O.

Example 3.8: An object of mass 50 g moves uniformly along a circular orbit with an angular speed of 5 rad/s. If the linear speed of the particle is 25 m/s, what is the radius of the circle? Calculate the centripetal force acting on the particle.

Do you know ?

1. The centripetal force is not one of the external forces acting on the object. As can be seen from above, the actual forces acting on the bob are T and mg, the resultant of these is the centripetal force. Conversely, if the resultant force is centripetal, motion must be circular.
2. In planetary motion, the gravitational force between Sun and the planets provides the necessary centripetal force for the circular motion.

Solution: The linear speed and angular speed are related by $v = \omega r$

$\therefore\ r = v/\omega = 25/5\ m = 5\ m.$

Centripetal force acting on the object $= \frac{mv^2}{r} = \frac{0.05 \times 25^2}{5} = 6.25$ N.

Example 3.9: An object is travelling in a horizontal circle with uniform speed. At $t = 0$, the velocity is given by $\vec{u} = 20\hat{i} + 35\hat{j}$ km/s. After one minute the velocity becomes $\vec{v} = -20\hat{i} - 35\hat{j}$. What is the magnitude of the acceleration?

Solution: Magnitude of initial and final velocities =

$$= u = \sqrt{(20)^2 + (35)^2}\ \text{m/s}$$

$$= \sqrt{1625}\ \text{m/s}$$

$$= 40.3\ \text{m/s}$$

As the velocity reverses in 1 min, the time period of revolution is 2 min.

$$T = \frac{2\pi r}{u},\ \text{giving}\ r = \frac{uT}{2\pi}$$

$$a = \frac{u^2}{r} = \frac{u^2 2\pi}{uT} = \frac{2\pi u}{T} = \frac{\cancel{2} \times 3.14 \times 40.3}{\cancel{2} \times 60}$$

$$= 2.11\ \text{m s}^{-2}$$

Internet my friend

1. hyperphysics.phy-astr.gsu.edu/hbase/mot.html#motcon
2. www.college-physics.com/book/mechanics

Exercises

1. Choose the correct option.

i) An object thrown from a moving bus is on example of
   (A) Uniform circular motion
   (B) Rectilinear motion
   (C) Projectile motion
   (D) Motion in one dimension

ii) For a particle having a uniform circular motion, which of the following is constant
   (A) Speed (B) Acceleration
   (C) Velocity (D) Displacement

iii) The bob of a conical pendulum under goes
   (A) Rectilinear motion in horizontal plane
   (B) Uniform motion in a horizontal circle
   (C) Uniform motion in a vertical circle
   (D) Rectilinear motion in vertical circle

iv) For uniform acceleration in rectilinear motion which of the following is not correct?
   (A) Velocity-time graph is linear
   (B) Acceleration is the slope of velocity time graph
   (C) The area under the velocity-time graph equals displacement
   (D) Velocity-time graph is nonlinear

v) If three particles A, B and C are having velocities $\vec{v}_A$, $\vec{v}_B$ and $\vec{v}_C$ which of the following formula gives the relative velocity of A with respect to B
   (A) $\vec{v}_A + \vec{v}_B$ (B) $\vec{v}_A - \vec{v}_C + \vec{v}_B$

(C) $\vec{v}_A - \vec{v}_B$ (D) $\vec{v}_C - \vec{v}_A$

2. Answer the following questions.

i) Separate the following in groups of scalar and vectors: velocity, speed, displacement, work done, force, power, energy, acceleration, electric charge, angular velocity.

ii) Define average velocity and instantaneous velocity. When are they same?

iii) Define free fall.

iv) If the motion of an object is described by $x = f(t)$ write formulae for instantaneous velocity and acceleration.

v) Derive equations of motion for a particle moving in a plane and show that the motion can be resolved in two independent motions in mutually perpendicular directions.

vi) Derive equations of motion graphically for a particle having uniform acceleration, moving along a straight line.

vii) Derive the formula for the range and maximum height achieved by a projectile thrown from the origin with initial velocity $\vec{u}$ at an angel θ to the horizontal.

viii) Show that the path of a projectile is a parabola.

ix) What is a conical pendulum? Show that its time period is given by $2\pi\sqrt{\frac{l\cos\theta}{g}}$, where l is the length of the string, θ is the angle that the string makes with the vertical and g is the acceleration due to gravity.

x) Define angular velocity. Show that the centripetal force on a particle undergoing uniform circular motion is $-m\omega^2\vec{r}$.

3. Solve the following problems.

i) An aeroplane has a run of 500 m to take off from the runway. It starts from rest and moves with constant acceleration to cover the runway in 30 sec. What is the velocity of the aeroplane at the take off ?

[Ans: 120 km/hr]

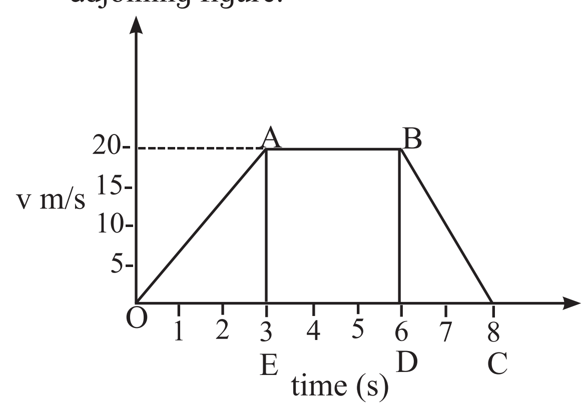
ii). A car moving along a straight road with a speed of 120 km/hr, is brought to rest by applying brakes. The car covers a distance of 100 m before it stops. Calculate (i) the average retardation of the car (ii) time taken by the car to come to rest.

[Ans: 50/9 m/sec^2, 6 sec]

iii) A car travels at a speed of 50 km/hr for 30 minutes, at 30 km/hr for next 15 minutes and then 70 km/hr for next 45 minutes. What is the average speed of the car?

[Ans: 56.66 km/hr]

iv) A velocity-time graph is shown in the adjoining figure.



Determine:

(i) initial speed of the car (ii) maximum speed attained by the car (iii) part of the graph showing zero acceleration (iv) part of the graph showing constant retardation (v) distance travelled by the car in first 6 sec.

[Ans: (i) 0 (ii) 20 m/sec (iii) AB (iv) BC (v) 90 m]

v) A man throws a ball to maximum horizontal distance of 80 meters. Calculate the maximum height reached.

[Ans: 20 m]

vi) A particle is projected with speed v_0 at angle θ to the horizontal on an inclined surface making an angle ϕ ($\phi < \theta$) to the horizontal. Find the range of the projectile along the inclined surface.

[Ans: $R = \frac{2v_0^2\cos\theta\sin(\theta-\phi)}{g\cos^2\phi}$]

vii) A metro train runs from station A to B to C. It takes 4 minutes in travelling from station A to station B. The train halts at station B for 20 s. Then it starts from station B and reaches station C in next 3 minutes. At the start, the train accelerates for 10 sec to reach the constant speed of 72 km/hr. The train moving at the constant speed is brought to rest in 10 sec. at next station. (i) Plot the velocity- time graph for the train travelling from the station A to B to C. (ii) Calculate the distance between the stations A, B and C.

[Ans: AB = 4.6 km, BC =3.4 km]

viii) A train is moving eastward at 10 m/sec. A waiter is walking eastward at 1.2m/sec; and a fly is flying toward the north across the waiter's tray at 2 m/s. What is the velocity of the fly relative to Earth

[Ans: 11.4 m/s, 10° due north of east]

ix) A car moves in a circle at the constant speed of 50 m/s and completes one revolution in 40 s. Determine the magnitude of acceleration of the car.

[Ans: 7.85 m s^{-2}]

x) A particle moves in a circle with constant speed of 15 m/s. The radius of the circle is 2 m. Determine the centripetal acceleration of the particle.

[Ans: 112.5 m s^{-2}]

xi) A projectile is thrown at an angle of 30° to the horizontal. What should be the range of initial velocity (u) so that its range will be between 40m and 50 m? Assume $g = 10$ m s^{-2}.

[Ans: $21.49 \le u \le 24.03$ m s^{-2}]

***

4. Laws of Motion

Can you recall?

1. What are different types of motions?
2. What do you mean by kinematical equations and what are they?
3. Newton's laws of motion apply to most bodies we come across in our daily lives.
4. All bodies are governed by Newton's law of gravitation. Gravitation of the Earth results into weight of objects.
5. Acceleration is directly proportional to force for fixed mass of an object.
6. Bodies possess potential energy and kinetic energy due to their position and motion respectively which may change. Their total energy is conserved in absence of any external force.

4.1. Introduction:

If an object continuously changes its position, it is said to be in motion. Mechanics is a branch of Physics that deals with motion. There are basically two branches of mechanics (i) Statics, where we deal with objects at rest or in equilibrium under the action of balanced forces and (ii) Kinetics, which deals with actual motion.

Kinetics can be further divided into two branches **(i) Kinematics:** In kinematics, we describe various motions without discussing their cause. Various parameters discussed in kinematics are distance, displacement, speed, velocity and acceleration. **(ii) Dynamics:** In dynamics we describe the motion along with its cause, which is force and/or torque. Parameters discussed in dynamics are momentum, force, energy, power, etc. in addition to those in kinematics.

It must be understood that motion is strictly a relative concept, i.e., it should always be described in context to a reference frame. For example, if you are in a running bus, neither you nor your co-passengers sitting in the bus are in motion in your reference, i.e., moving bus. However, from the ground reference, bus, you and all the passengers are in motion.

If not random, motions in real life may be understood separately as linear, circular or rotational, oscillatory, etc., or some combinations of these. While describing any of these, we need to know the corresponding forces responsible for these motions. Trajectory of any motion is decided by acceleration $\vec{a}$ and the initial velocity $\vec{u}$.

a) Linear motion: Initial velocity may be zero or non-zero. If initial velocity is zero (starting from rest), acceleration in any direction will result into a linear motion.

If initial velocity is not zero, the acceleration must be in line with the initial velocity (along the same or opposite direction to that of the initial velocity) for resultant motion to be linear.

b) Circular motion: If initial velocity is not zero and acceleration is throughout perpendicular to the velocity, the resultant motion will be circular.

c) Parabolic motion: If acceleration is constant and initial velocity is *not* in line with the acceleration, the motion is parabolic, e.g., trajectory of a projectile motion.

d) Other combinations of $\vec{u}$ and $\vec{a}$ will result into different more complicated motion.

4.2. Aristotle's Fallacy:

Aristotle (384BC-322BC) stated that "an external force is required to keep a body in uniform motion". This was probably based on a common experience like a ball rolling on a surface stops after rolling through some distance. Thus, to keep the ball moving with constant velocity, we have to continuously apply a force on it. Similar examples can be found elsewhere, like a paper plane flying through air or a paper boat propelled with some initial velocity.

Correct explanation to Aristotle's fallacy was first given by Galileo (1564-1642), which was later used by Newton (1643-1727) in formulating *laws of motion*. Galileo showed that all the objects stop moving because of some resistive or opposing forces like friction, viscous drag, etc. In these examples such forces are frictional force for rolling ball, viscous drag or viscous force of air for paper plane and viscous force of water for the boat.

Thus, in reality, for an uninterrupted motion of a body an additional external force is required for overcoming these opposing forces.

Can you tell?

1. Was Aristotle correct?
2. If correct, explain his statement with an illustration.
3. If wrong, give the correct modified version of his statement.

4.3. Newton's Laws of Motion:

First law: Every inanimate object continues to be in its state of rest or of uniform *unaccelerated* motion unless and until it is acted upon by an *external, unbalanced* force.

Second law: Rate of change of linear momentum of a rigid body is directly proportional to the applied force and takes place in the direction of the applied force. On selecting suitable units, it takes the form $\vec{F} = \frac{d\vec{p}}{dt}$ (where $\vec{F}$ is the force and $\vec{p} = m\vec{v}$ is the linear momentum.

Third law: To every action (force), there is an equal and opposite reaction (force).

Discussion: From Newton's second law of motion, $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$. For a *given* body, mass m is constant.

$$\therefore \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \ldots \text{(for constant mass)}$$

Thus, if $\vec{F} = 0$, $\vec{v}$ is constant. Hence if there is no force, velocity will not change. This is nothing but Newton's first law of motion.

Can you tell?

What is then special about Newton's first law if it is derivable from Newton's second law?

4.3.1. Importance of Newton's First Law of Motion:

(i) It shows an equivalence between '*state of rest*' and '*state of uniform motion along a straight line*' as both need a net unbalanced force to change the state. Both these are referred to as '*state of motion*'. The distinction between state of rest and uniform motion lies in the choice of the '*frame of reference*'.

(ii) It defines *force* as an entity (or a physical quantity) that brings about a change in the '*state of motion*' of a body, i.e., force is something that initiates a motion or controls a motion. Second law gives its quantitative understanding or its mathematical expression.

(iii) It defines *inertia* as a fundamental property of every physical object by which the object *resists* any change in its state of motion. Inertia is measured as the mass of the object. More specifically it is called inertial mass, which is the ratio of net force ($|\vec{F}|$) to the corresponding acceleration ($|\vec{a}|$).

4.3.2. Importance of Newton's Second Law of Motion:

(i) It gives mathematical formulation for quantitative measure of force as rate of change of linear momentum.

Do you know ?

Mathematical expression for force must be remembered as $\vec{F} = \frac{d\vec{p}}{dt}$ and ***not*** as $\vec{F} = m\vec{a}$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{dm}{dt}(\vec{v}) + m\left(\frac{d\vec{v}}{dt}\right)$$

$$= \frac{dm}{dt}(\vec{v}) + m\vec{a}$$

For a *given* body, mass is constant, i.e., $\frac{dm}{dt} = 0$ and only in this case, $\vec{F} = m\vec{a}$

In the case of a rocket, both the terms are needed as both mass and velocity are varying.

(ii) It defines *momentum* $\left(\vec{p}=m\vec{\text{v}}\right)$ instead of velocity as the fundamental quantity related to motion. What is changed by a force is the momentum and not necessarily the velocity.

(iii) Aristotle's fallacy is overcome by considering *resultant unbalanced* force.

4.3.3 Importance of Newton's Third Law of Motion:

(i) It defines *action* and *reaction* as a pair of equal and opposite forces acting along the same line.

(ii) Action and reaction forces are always on *different* objects.

Consequences:

Action force exerted by a body *x* on body *y*, conventionally written as $\vec{F}_{yx}$, is the force experienced by *y*.

As a result, body *y* exerts *reaction force* $\vec{F}_{xy}$ on body *x*.

In this case, body *x* experiences the force $\vec{F}_{xy}$ only while the body *y* experiences the force $\vec{F}_{yx}$ only.

Forces $\vec{F}_{xy}$ and $\vec{F}_{yx}$ are equal in magnitude and opposite in their directions, but there is no question of cancellation of these forces as those are experienced by *different* objects.

Forces $\vec{F}_{xy}$ and $\vec{F}_{yx}$ *need not* be contact forces. Repulsive forces between two magnets is a pair of action-reaction forces. In this case the two magnets are not in contact. Gravitational force between Earth and moon or between Earth and Sun are also similar pairs of non-contact action-reaction forces.

Example 4.1: A hose pipe used for gardening is ejecting water horizontally at the rate of 0.5 m/s. Area of the bore of the pipe is 10 cm^2. Calculate the force to be applied by the gardener to hold the pipe *horizontally* stationary.

Solution: If ejecting water horizontally is considered as action force on the water, the water exerts a backward force (called recoil force) on the pipe as the reaction force.

$$\vec{F}=\frac{d\vec{p}}{dt}=\frac{d\left(m\vec{\text{v}}\right)}{dt}=\frac{dm}{dt}\vec{\text{v}}+m\frac{d\vec{\text{v}}}{dt}$$

As v, the velocity of ejected water is constant, $F=\frac{dm}{dt}\vec{\text{v}}$, where $\frac{dm}{dt}$ is the rate at which mass of water is ejected by the pipe.

As the force is in the direction of velocity (horizontal), we can use scalars. $\therefore F=\frac{dm}{dt}\text{v}$

$$\frac{dm}{dt}=\frac{d\left(V\rho\right)}{dt}=\frac{d\left(Al\rho\right)}{dt}=A\rho\frac{dl}{dt}=A\rho\text{v}$$

where V = volume of water ejected

A = area of cross section of bore = 10 cm^2

ρ = density of water = 1 g/cc

l = length of the water ejected in time t

$\frac{dl}{dt}=\text{v}=$ velocity of water ejected

$=0.5$ m/s $=50$ cm/s

$$F=\frac{dm}{dt}\text{v}=\left(A\rho\text{v}\right)\text{v}=A\rho\text{v}^2=10\times1\times50^2$$

$$=25000\text{ dyne}=0.25\text{ N}$$

Equal and opposite force must be applied by the gardener.

4.4. Inertial and Non-Inertial Frames of Reference

Consider yourself standing on a railway platform or a bus stand and you see a train or bus moving. According to you, that train or bus is moving or is in motion. As per the experience of the passengers in the train or bus, they are at rest and you are moving (in backward direction). Hence motion itself is a relative concept. To know or describe a motion you need to describe or define some reference. Such a reference is called a *frame of reference*. In the example discussed above, if you consider the platform as the reference, then the passengers and the train are moving. However, if the train is considered as the reference, you and platform, etc. are moving.

Usually a set of coordinates with a suitable origin is enough to describe a frame of reference. If position coordinates of an object are continuously changing with time in a frame of reference, then *that object* is in motion in *that* frame of reference. Any frame of reference in which Newton's first law of motion is applicable is the simplest understanding of an inertial frame of reference. It means, if there is no net force, there is no acceleration. Thus in an inertial frame, a body will move with constant velocity (which may be zero also) if there is no net force acting upon it. In the absence of a net force, if an object suffers an acceleration, that frame of reference is ***not*** an inertial frame and is called as non-inertial frame of reference.

Measurements in one inertial frame can be converted to measurements in another inertial frame by a simple transformation, i.e., by simply using some velocity vectors (relative velocity between the two frames of reference).

Illustration: Imagine yourself inside a car with all windows opaque so that you can not see anything outside. Also consider that there is a pendulum tied inside the car and not set into oscillations . If the car just starts its motion (with reference to outside or ground), you will experience a jerk, i.e., acceleration inside the car even though there is no force acting upon you. During this time, the string of the pendulum may be steady, but ***not*** vertical. During time of acceleration, the car can be considered to be a non-inertial frame of reference. Later on if the car is moving with constant velocity (with reference to the ground), you will not experience any jerky motion within the car and the car can be considered as an inertial frame of reference. In this case, the pendulum string will be vertical, when not oscillating.

Do you know ?

The situations/phenomena that can be explained using Newton's laws of motion fall under Newtonian mechanics. So far as our daily life situations are considered, Newtonian mechanics is perfectly applicable. However, under several extreme conditions we need to use some other theories.

Limitation of Newton's laws of motion

(i) Newton's laws are applicable only in the inertial frames of reference (discussed later). If the body is in a frame of reference of acceleration (a), we need to use a *pseudo* force $\left(-m\vec{a}\right)$ in addition to all the other forces while writing the force equations.

(ii) Newton's laws are applicable for *point* objects.

(iii) Newton's laws are applicable to rigid bodies. A body is said to be *rigid* if the relative distances between its particles do not change for any deforming force.

(iv) For objects moving with speeds comparable to that of light, Newton's laws of motion do not give results that match with the experimental results and Einstein's special theory of relativity has to be used.

(v) Behaviour and interaction of objects having atomic or molecular sizes cannot be explained using Newton's laws of motion, and quantum mechanics has to be used.

A rocket in intergalactic space (gravity free space between galaxies) with all its engines shut is closest to an ideal inertial frame. However, Earth's acceleration in the reference frame of the Sun is so small that any frame attached to the Earth can be used as an inertial frame for any day-to-day situation or in our laboratories.

4.5 Types of Forces:

4.5.1. Fundamental Forces in Nature:

All the forces in nature are classified into following **four** interactions that are termed as fundamental forces.

(i) Gravitational force: It is the attractive force between two (point) masses separated by a distance. Magnitude of gravitational force between point masses m_1 and m_2 separated by distance r is given by $F = \dfrac{Gm_1m_2}{r^2}$

where G = 6.67×10^{-11} SI units. Between two point masses (particles) separated by a given distance, this is the weakest force having infinite range. This force is always attractive. Structure of the universe is governed by this force.

Common experience of this force for us is gravitational force exerted by Earth on us, which we call as our weight W.

$$\therefore W = \frac{GMm}{R^2} = m\left(\frac{GM}{R^2}\right) = mg$$,

where M and R represent respectivily mass and radius of the Earth. Distance between ourselves and Earth is taken as radius of the Earth when we are on the surface of the Earth because our size is negligible as compared to radius of the Earth (6.4×10^6 m).

$$\left(\frac{GM}{R^2}\right) = \frac{6.67\times10^{-11}\times6\times10^{24}}{\left(6.4\times10^6\right)^2}$$

$\cong 9.8$ m/s^2 = g = gravitational acceleration or gravitational field intensity.

We ***feel*** this force only due to normal reaction from the surface of our contact with Earth.

All individual bodies also exert gravitation force on each other but it is too small compared to that by the Earth. For example, mutual gravitational force between two SUMO wrestlers, each of mass 300 kg, assuming the distance between them is 0.5 m, will be

$$F = \frac{6.67\times10^{-11}\times300\times300}{0.5^2}$$

$$\cong 24\times10^{-5}\,\mathrm{N} = 24\ \mu\mathrm{N}.$$

This force is negligibly small in comparison to the weight of each SUMO wrestler $\cong$ 3000 N.

(ii) Electromagnetic (EM) force: It is an attractive or repulsive force between electrically charged particles. Earlier, electric and magnetic forces were thought to be independent. After the demonstrations by Michael Faraday (1791-1867) and James Clerk Maxwell (1831-1879), electric and magnetic forces were unified through the theory of electromagnetism. These forces are stronger than the gravitational force. Our life is practically governed by these forces. Majority of forces experienced in our daily life, such as force of friction, normal reaction, tension in strings, collision forces, elastic forces, viscosity (fluid friction), etc. are EM in nature. Under the action of these forces, there is *deformation of objects* that changes intermolecular distances thereby resulting into reaction forces.

(iii) Strong (nuclear) force: This is the strongest force that binds the nucleons together inside a nucleus. Though strongest, it is a short range ($< 10^{-14}$ m) force. Therefore is very strong attractive force and is charge independent.

(iv) Weak (nuclear) force: This is the interaction between subatomic particles that is responsible for the radioactive decay of atoms, in particular beta emission. The weak nuclear force is not as weak as the gravitational force, but much weaker than the strong nuclear and EM forces. The range of weak nuclear force is exceedingly small, of the order of 10^{-16} m.

Weak interaction force:

The radioactive isotope C^{13} is converted into N^{14} in which a neutron is converted into a proton. This property is used in *carbon dating* to determine the age of a sample.

In radioactive beta decay, the nucleus emits an electron (or positron) and an uncharged particle called neutrino. There are two types of β-decay, β^+ and β^-. During β^+decay, a proton is converted into a neutron (accompanied by positron emission) and during β^- decay a neutron is converted into a proton (accompanied by electron emission).

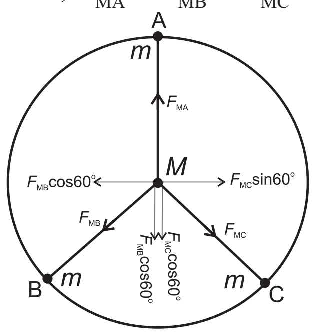
Another most interesting illustration of weak forces is fusion reaction in the core of the Sun. During this, protons are converted into neutrons and a neutrino is emitted due to energy balance. In general, emission of a neutrino is the evidence that there is conversion of a proton into a neutron or a neutron into a proton. This is possible *only* due to weak forces.

Example 4.2. Three identical point masses are fixed symmetrically on the periphery of a circle. Obtain the resultant gravitational force on any point mass M at the centre of the circle. Extend this idea to more than three identical masses symmetrically located on the periphery. How far can you extend this concept?

Solution:

(i) Figure below shows three identical point masses m on the periphery of a circle of radius r. Mass M is at the centre of the circle. Gravitational forces on M due to these masses are attractive and are directed as shown.

In magnitude, $F_{MA} = F_{MB} = F_{MC} = \dfrac{GMm}{r^2}$



Forces F_{MB} and F_{MC} are resolved along F_{MA} and perpendicular to F_{MA} as shown. Components perpendicular to F_{MA} cancel each other. Components along F_{MA} are $F_{MB}\cos 60^0 = F_{MC}\cos 60^0 = \frac{1}{2}F_{MA}$ each. Magnitude of their resultant is F_{MA} and its direction is opposite to that of F_{MA}. Thus, the resultant force on mass M is zero.

(ii) For any even number of equal masses, the force due to any mass m is balanced (cancelled) by diametrically opposite mass. For any odd number of masses, as seen for 3, the components perpendicular to one of them cancel each other while the components parallel to one of these add up in such a way that the resultant is zero for any number of identical masses m located symmetrically on the periphery.

(iii) As the number of masses tends to infinity, their collective shape approaches circumference of the circle, which is nothing but a ring. Thus, the gravitational force exerted by a ring mass on any other mass at its centre is zero.

In three-dimensions, we can imagine a uniform hollow sphere to be made up of infinite number of such rings with a common diameter. Thus, the gravitational force for any mass kept at the centre of a hollow sphere is zero.

Do you know ?

Unification of forces: Newton unified terrestrial (related to Earth and hence to our daily life) and celestial (related to universe) domains under a common law of gravitation. The experimental discoveries of Oersted (1777-1851) and Faraday showed that electric and magnetic phenomena are in general inseparable leading to what is called 'EM phenomenon'. Electromagnetism and optics were unified by Maxwell with the proposition that light is an EM wave. Einstein attempted to unify gravity and electromagnetism under general relativity but could not succeed. The EM and the weak nuclear force have now been unified as a single 'electro-weak' force.

4.5.2. Contact and Non-Contact Forces:

For some forces, like gravitational force, electrostatic force, magnetostatic force, etc., physical contact is not an essential condition. These forces exist even if the objects are distant or physically separated. Such forces are non-contact forces.

Forces resulting *only* due to contact are called contact forces. All these are EM in nature, arising due to some deformation. Normal reaction, forces occurring during collision, force of friction, etc., are contact forces. There are two common categories of contact forces. Two objects in contact, while exerting mutual force, try to push each other away along their common normal. Quite often we call it as 'normal reaction' force or 'normal' force. While standing on a table, we *push* the table away from us (downward) and the table *pushes* us away from it (upward) both being equal in magnitude and acting along the same '*normal*' line.

Force of friction is also a contact force that arises whenever there is a relative motion or tendency of relative motion between surfaces in contact. This is the parallel (or tangential) component of the reaction force. In this case, the molecules of surfaces in contact, which have developed certain equilibrium, are required to be separated.

4.5.3 Real and Pseudo Forces:

Consider ourselves inside a lift (or elevator). When the lift just starts moving up (accelerates upward), we feel a bit heavier as if someone is pushing us down. This is not imaginary or not just a feeling. If we are standing on a weighing scale inside this lift, during this time the weighing scale records an increase in weight. During travelling with uniform upward velocity no such change is recorded. While stopping at some upper floor, the lift undergoes downward acceleration for decreasing the upward velocity. In this case the weighing scale records loss in weight and we also feel lighter. These *extra* upward or downward forces are (i) Measurable, means they are not imaginary, (ii) not accountable as per Newton's second law in the inertial frame and (iii) not among any of the four fundamental forces.

When we are inside a bus such forces are experienced when the bus starts to move (forward acceleration), when the bus is about to stop (backward acceleration) or takes a turn (centripetal acceleration). In all these cases we are inside an accelerated system (which is our frame of reference). If a force measuring device is suitably used – like the weighing scale recording the change in weight – these forces can be recorded and will be found to be always opposite to the acceleration of your frame of reference. They are also exactly equal to -$m\vec{a}$, where m is our mass and $\vec{a}$ is acceleration of the system (frame of reference).

We have already defined or described real forces to be those which obey Newton's laws of motion and are one of the four fundamental forces. Forces in above illustrations do not satisfy this description and cannot be called real forces. Hence these are called ***pseudo*** forces. ***Pseudo*** in this case does not mean imaginary (because these are measurable with instruments) but just means *non-real*. These forces are measured to be $(-m\vec{a})$. Hence, a term $(-m\vec{a})$ added to resultant force enables us to apply Newton's laws of motion to a non-inertial frame of reference of acceleration $\vec{a}$. Negative sign refers to their direction, which is opposite to that of the acceleration of the reference frame.

As per the illustration of the lift with downward acceleration $\vec{a}$, the weight experienced will be $\vec{W} = m\vec{g} + (-m\vec{a})$

As $\vec{g}$ and $\vec{a}$ are along the same direction in this case, $W = mg - ma$. This explains the *feeling* of a loss in weight.

During upward acceleration, say $\vec{a_1}$, we have, $\vec{W_1} = m\vec{g} + (+m\vec{a}_1)$

In this case, $\vec{g}$ and $\vec{a_1}$ are oppositely directed. $\therefore W_1 = mg + (+ma_1) = mg + ma_1$ that explains gain in weight or existence of extra downward force.

In mathematics we define a number to be real if its square is zero or positive. Solution set of equations like x^2 - $6x$ + 10 = 0 does not satisfy the criterion to be a real number. Such numbers are *complex* numbers which include $i = \sqrt{-1}$ along with some real part. It means every *non-real* need not be *imaginary* as in literal verbal sense.

Example 4.3: A car of mass 1.5 ton is running at 72 kmph on a straight horizontal road. On turning the engine off, it stops in 20 seconds. While running at the same speed, on the same road, the driver observes an accident 50 m in front of him. He immediately applies the brakes and just manages to stop the car at the accident spot. Calculate the braking force.

Solution: On turning the engine off,

$u = 20\,\mathrm{m\ s^{-1}}$, v = 0, $t = 20\,s$

$$\therefore a = \frac{\mathrm{v} - u}{t} = -1\,\mathrm{m\ s^{-2}}$$

This is frictional retardation (negative acceleration).

After seeing the accident,

$u = 20\,\text{m s}^{-1}$, $v = 0$, $s = 50\,\text{m}$

$$\therefore a_1 = \frac{v^2 - u^2}{2s} = -4\,\text{m s}^{-2}$$

This retardation is the combined effect of braking and friction. Thus, braking retardation $= 4 - 1 = 3\,\text{m s}^{-2}$.

$\therefore$ Braking force = mass × braking retardation = 1500 × 3 = 4500 N

4.5.4. Conservative and Non-Conservative Forces and Concept of Potential Energy:

Consider an object lying on the ground is lifted and kept on a table. Neglecting air resistance, the amount of work done is the work done *against* gravitational force and it is *independent* of the actual path chosen (Remember, as there is no air resistance, gravitational force is the only force). Similarly, while keeping the same object back on the ground from the table, the work is done ***by*** the gravitational force. In either case the amount of work done is the same ***and*** is independent of the actual path chosen. The work done by force $\vec{F}$ in moving the object through a distance dx can be mathematically represented as $dW = \vec{F}.d\vec{x}$ =-dU or $dU = -\vec{F}.d\vec{x}$.

If work done by or against a force is independent of the actual path, the force is said to be a ***conservative force***. During the work done by a conservative force, the mechanical energy (sum of kinetic and potential energy) is conserved. In fact, we define the concept of potential at a point or potential energy (in the topic of gravitation) with the help of conservative forces only. The work done by or against conservative forces reflects an equal amount of change in the *potential energy*. The corresponding work done is used in changing the position or in achieving the new position in the gravitational field. Hence, potential energy is often referred to as the energy possessed on account of position.

In the illustration given above, the work done is reflected as increase in the gravitational potential energy when the displacement is *against* the (gravitational) force. Same amount of potential energy is decreased when the displacement is in the direction of force. In either case it is independent of the actual path but depends only upon the initial and final positions. This change in the potential energy takes place in such a way that the mechanical energy is conserved.

As discussed above, increase in the potential energy, $dU = -\vec{F}.d\vec{x}$ or $U = -\int \vec{F}.d\vec{x}$ where $\vec{F}$ is a conservative force. This concept, will be described in details in Chapter 5 on Gravitation in context of gravitational potential energy and gravitational potential.

During this process, if friction or air resistance is present, additional work is necessary against the frictional force (for the same displacement). This work is strictly path dependent and not recoverable. Such forces (like friction, air drag, etc.) are called *non-conservative* forces. Work done against these forces appears as heat, sound, light, etc. The work done against non-conservative forces is not recoverable even if the path is exactly reversed.

4.5.5. Work Done by a Variable Force:

The popular formula for calculating work done is $W = \vec{F}\cdot\vec{s} = F\,s\cos\theta$ where θ is the angle between the applied force $\vec{F}$ and the displacement $\vec{s}$.

This formula is applicable only if both force $\vec{F}$ and displacement $\vec{s}$ are constant and finite. In several real-life situations, the force is not constant. For example, while lifting an object through several thousand kilometres, the gravitational force is not constant. The viscous forces like fluid resistance depend upon the speed, hence, quite often are not constant with time. In order to calculate the work done by such variable forces we use integration.

Illustration: Figure 4.1(a) shows variation of a force $\vec{F}$ plotted against corresponding displacements in its direction $\vec{s}$. As the displacement is in the direction of the applied force, vector nature is not used. We need to calculate the work done by this force during displacement from s_1 to s_2. As the force is variable, using $W = F(s_2 - s_1)$ directly is not possible. In order to use integration, let us divide the displacement into a large number of infinitesimal (infinitely small) displacements. One of such displacements is *ds*. It is so small that the force *F* is practically constant for this displacement. Practically constant means the change in the force is so small that the change can not be recorded. The shaded strip shows one of such displacements. As the force is constant, the area of this strip *F.ds* is the work done *dW* for this displacement. Total work done *W* for displacement $(s_2 - s_1)$ can then be obtained by using integration.

$$\therefore W = \int_{s_1}^{s_2} F.ds$$

Method of integration is applicable if the exact way of variation in $\vec{F}$ with $\vec{s}$ is known and that function is integrable.

The area under the curve between s_1 and s_2 also gives the work done *W* (if the force axis necessarily starts with zero), as it consists of all the strips of *ds* between s_1 and s_2. In Fig. 4.1(b), the variation in the force is linear. In this case, the area of the trapezium AS_1S_2B gives total work done *W*.

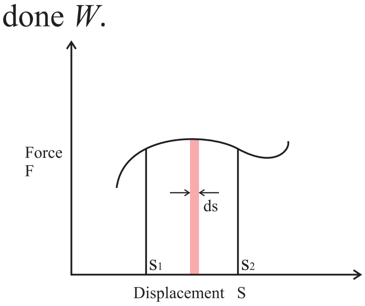


Fig 4.1 (a): Work done by nonlinearly varying force.

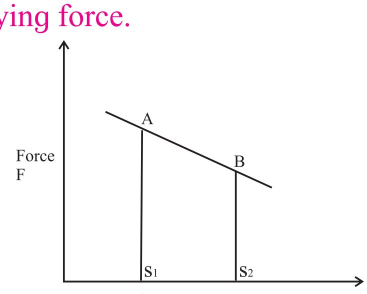


Fig 4.1 (b): Work done by linearly varying force.

Example 4.4: Over a given region, a force (in newton) varies as $F = 3x^2 - 2x + 1$. In this region, an object is displaced from $x_1 = 20$ cm to $x_2 = 40$ cm by the given force. Calculate the amount of work done.

Solution:

$$W = \int_{s_1}^{s_2} F.ds = \int_{0.2}^{0.4} (3x^2 - 2x + 1)dx$$

$$= \left[x^3 - x^2 + x\right]_{0.2}^{0.4}$$

$$= \left[0.4^3 - 0.4^2 + 0.4\right] - \left[0.2^3 - 0.2^2 + 0.2\right]$$

$$= 0.304 - 0.168 = 0.136\,\text{J}$$

4.6. Work Energy Theorem:

If there is a decrease in the potential energy (like a body falling down) due to a conservative force, it is entirely converted into kinetic energy. Work done by the force then appears as kinetic energy. Vice versa if an object is moving against a conservative force its kinetic energy decreases by an amount equal to the work done against the force. This principle is called *work-energy theorem* for conservative forces.

Case I: Consider an object of mass *m* moving with velocity $\vec{u}$ experiencing a constant opposing force $\vec{F}$ which slows it down to $\vec{v}$ during displacement $\vec{s}$. As $\vec{u}$ and $\vec{F}$ are oppositely directed, the entire motion will be along the same *line*. In this case we need not use the vector form, just ± signs should be good enough.

If $a = \dfrac{F}{m}$ is the acceleration, we can write the relevant equation of motion as $v^2 - u^2 = 2(-a)s$ (negative acceleration for opposing force)

Multiplying throughout by m/2, we get

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = (ma).s = F.s$$

Left-hand side is decrease in the kinetic energy and the right-hand side is the work done by the force. Thus, change in kinetic energy is equal to work done by the conservative force, which is in accordance with work-energy theorem.

Case II: Accelerating conservative force along with a retarding non-conservative force:

An object dropped from some point at height h falls down through air. While coming down its potential energy decreases. Equal amount of work is done in this case also. However, this time the work is not entirely converted into kinetic energy but some part of it is used in overcoming the air resistance. This part of energy appears in some other forms such as heat, sound, etc. In this case, the work-energy theorem can mathematically be written as $\Delta PE = \Delta KE + W_{air\ resistance}$

(Decrease in the gravitational P.E. = Increase in the kinetic energy + work done against non-conservative forces). Magnitude of air resistance force is not constant but depends upon the speed hence it can be written as $\int \vec{F}.\vec{ds}$ as seen during work done by (or against) a variable force.

4.7. Principle of Conservation of Linear Momentum:

According to Newton's second law, resultant force is equal to the rate of change of linear momentum or $\vec{F} = \dfrac{d\vec{p}}{dt}$

In other words, if there is no resultant force, the linear momentum will not change or will remain constant or will be conserved. Mathematically, if $\vec{F} = 0, \dfrac{d\vec{p}}{dt} = 0$ or $\vec{p}$ is constant

Always remember

Isolated system means absence of any external force. A *system* refers to a set of particles, colliding objects, exploding objects, etc. *Interaction* refers to collision, explosion, etc. During any interaction among such objects the *total* linear momentum of the *entire system* of these particles/objects is constant. Remember, forces during collision or during explosion are *internal* forces for that *entire system*.

During collision of two particles, the two particles exert forces on *each other*. If these particles are discussed independently, these are external forces. However, for the *system of the two particles* together, these forces are internal forces.

(or conserved). This leads us to the *principle of conservation of linear momentum* which can be stated as **"The total momentum of an isolated system is conserved during any interaction."**

Systems and free body diagrams:

Mathematical approach for application of Newton's second law:

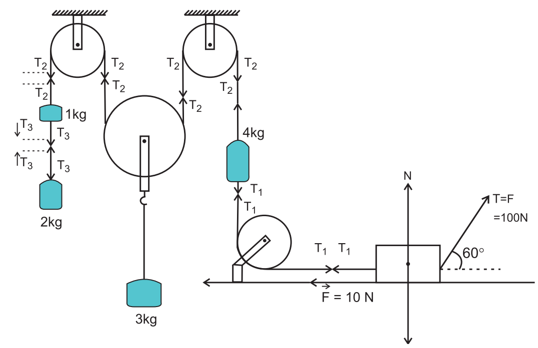


Fig 4.2 (a): System for illustration of free body diagram.

Consider the arrangement shown in Fig.4.2 (a). Pulleys P_1, P_3 and P_4 are fixed, while P_2 is movable. Force F = 100 N, applied at an angle 60° with the horizontal is responsible for the motion, if any. Contact surface of the 5 kg mass offers a constant opposing force F = 10 N. Except this, there are no resistive forces anywhere.

Discussion: Until 1 kg mass reaches the pulley P_1, the motion of 1 kg and 2 kg masses is identical. Thus, these two can be considered to be a *single system* of mass 3 kg except for knowing the tension T_3. The forces due to tension in the string joining them are internal forces for this system.

All masses *except* the 3 kg mass are travelling same distance in the same time. Thus, their accelerations, if any, have same magnitudes. If the string S connecting 1 kg and 4 kg masses moves by x, the lower string S_1 holding the 3 kg mass moves through $x/2$.

Free body diagrams (FBD): A free body diagram refers to forces acting on only one body at a time, and its acceleration.

Free body diagram of 2 kg mass: Let a be its upward acceleration. According to Newton's second law, it must be due to the resultant vertical force on this mass. To know this force, we need to know all the individual forces acting on this mass. The agencies exerting forces on this mass are Earth (downward force 1g) and force due to the tension T_3.

In this case, the lower half of the string is connected to the 2 kg mass. The direction of T_3 for lower part of the string is upwards as shown in the Fig. 4.2 (b). Upper part of the string is connected to the 1 kg mass. Thus, the direction of T_3 for 1 kg mass will be downwards. However, it will appear only for the free body diagram of the 1 kg mass and will not appear in the free body diagram of 2 kg mass. Hence, the free body diagram of the 2 kg mass will be as follows: Its force equation, according to Newton's second law will then be $T_3 - 2g = 2a$.

Practical tip: Easiest way to know the direction of forces due to tension is to put an X-mark on the string. Two halves of this cross indicate the directions of the forces exerted *by the string* on the bodies connected to either parts of the string.

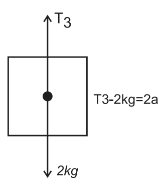


Fig 4.2 (b): Free body diagram for 2 kg mass.

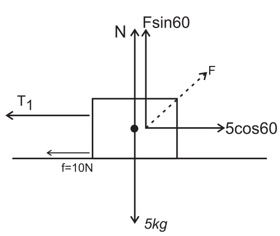


Fig 4.2 (c): Free body diagram for 5 kg mass.

Free body diagram of mass 5 kg: Its horizontal acceleration is also *a*, but towards right. The force exerting agencies are Earth (force 5g downwards), contact surface (normal force *N*, vertically upwards ***and*** opposing force *F* = 10 N, towards left), and the two strings on either side (Forces due to their tensions *T* and T_1). All these are shown in its free body diagrams in Fig. 4.2 (c). On resolving the force *F* along the vertical and horizontal directions, the free body diagram of 5 kg mass can be drawn as explained below.

As this mass has only horizontal motion, the vertical forces must cancel. Therefore, along the vertical direction,

$$N + F \sin 60° = 5g$$

Along the horizontal direction,

$$T_1 + 10(\text{opposing force}) = F\cos 60^0 = T \cos 60^0$$

Similar equations can be written for all the masses and also for the movable pully. On solving these equations simultaneously, we can obtain all the necessary quantities.

Example 4.5: Figure (a) shows a fixed pulley. A massless inextensible string with masses m_1 and $m_2 > m_1$ attached to its two ends is passing over the pulley. Such an arrangement is called an Atwood machine. Calculate accelerations of the masses and force due to the tension along the string assuming axle of the pulley to be frictionless.

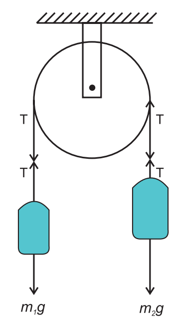


Fig. (a).

Solution: **Method I:** Direct method: As $m_2 > m_1$, mass m_2 is moving downwards and mass m_1 is moving upwards.

Net downward force

$$= F = (m_2)g - (m_1)g = (m_2 - m_1)g$$

As the string is inextensible, both the masses travel the same distance in the same time. Thus, their accelerations are numerically the same (one upward, other downward). Let it be *a*.

Thus, total mass in motion, $M = m_2 + m_1$

$$\therefore a = \frac{F}{M} = \left(\frac{m_2 - m_1}{m_2 + m_1}\right)g$$

For mass m_1, the upward force is the force due to tension *T* and downward force is *mg*. It has upward acceleration *a*. Thus, $T - m_1 g = ma$

$\therefore T = m_1(g + a)$

Using the expression for *a*, we get

$$T = \left(\frac{2m_1m_2}{m_1+m_2}\right)g$$

Method II: (Free body equations)

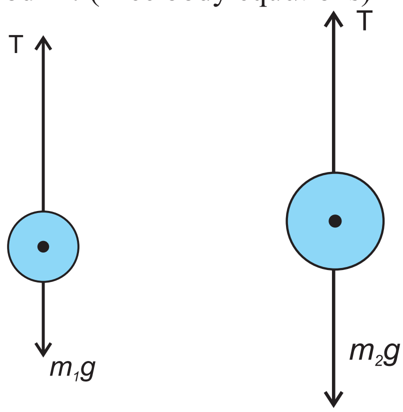


Fig. (b) Fig. (b)

Free body diagrams of m_1 and m_2 are as shown in Figs. (b) and (c).

Thus, for the first body, $T - m_1g = m_1a$ --- (I)

For the second body, $m_2g - T = m_2a$ --- (II)

Adding (I) and (II), and solving for a,

we get, $a = \left(\frac{m_2 - m_1}{m_2 + m_1}\right)g$ --- (III)

Solving Eqs. (II) and (III) for T, we get,

$$T = m_2(g - a) = \left(\frac{2m_1m_2}{m_1+m_2}\right)g$$

4.8. Collisions:

During collisions a number of objects come together, interact (exert forces on *each other*) and scatter in different directions.

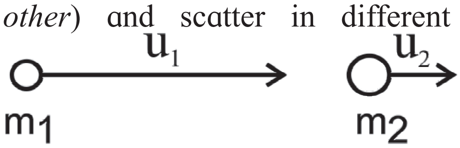


Fig. 4.3 (a): Head on collision-before collision.

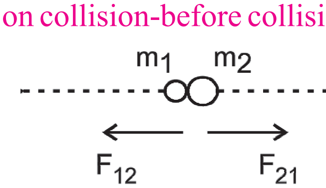


Fig. 4.3 (b): Head on collision-during impact.

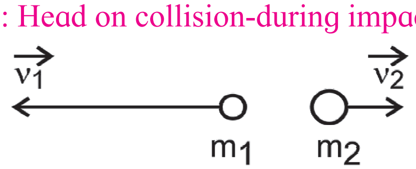


Fig. 4.3 (c): Head on collision-after collision.

4.8.1. Elastic and inelastic collisions:

During a collision, the linear momentum of the entire system of particles is always conserved as there is no external force acting on the *system* of particles. However, the individual momenta of the particles change due to mutual forces, which are internal forces.

$\therefore \sum \vec{p}_{initial} = \sum \vec{p}_{final}$, during ***any*** collision (or explosion), where $\vec{p}$'s are the linear momenta of the particles.

However, kinetic energy of the entire system may or may not conserve.

Collisions can be of two types: ***elastic collisions*** and ***inelastic collisions.***

Elastic collision: A collision is said to ***elastic*** if kinetic energy of the entire system is conserved during the collision (along with the linear momentum). Thus, during an elastic collision,

$$\sum K.E._{initial} = \sum K.E._{final}$$

An elastic collision is *impossible* in daily life. However, in many situations, the interatomic or intermolecular collisions are considered to be elastic (like in kinetic theory of gases, to be discussed in the next standard).

Inelastic Collision: A collision is said to be ***inelastic*** if there is a loss in the kinetic energy during collision, but linear momentum is conserved. The loss in kinetic energy is either due to internal friction or vibrational motion of atoms causing heating effect. Thus, during an inelastic collision,

$$\sum K.E._{initial} > \sum K.E._{final}.$$

During an explosion as energy is supplied internally. Thus,

$$\sum K.E._{final} > \sum K.E._{initial}.$$

As stated earlier, $\sum \vec{p}_{initial} = \sum \vec{p}_{final}$ for inelastic collisions or explosion also. In fact, this is always the first equation for discussing these interactions or while solving numerical questions.

4.8.2. Perfectly Inelastic Collision:

This is a special case of inelastic collisions. If colliding bodies join together after collision, it is said to be a ***perfectly inelastic collision***.

In other words, the colliding bodies have a common final velocity after a perfectly inelastic collision. Being an inelastic collision, obviously there is a loss in the kinetic energy of the system during a perfectly inelastic collision. In fact, the loss in kinetic energy is maximum in perfectly inelastic collision.

Illustrations:

(i) Consider a bullet fired towards a block kept on a smooth surface. Collision between bullet and the block will be elastic if the bullet rebounds with exactly the same initial speed and the block remains stationory. If the bullet gets embedded into the block and the two move jointly, it is perfectly inelastic collision. If the bullet rebounds with smaller speed or comes out of the block on the other side with some speed, it is an inelastic (or partially inelastic) collision. Remember, there is nothing called a partially elastic collision Elastic collisions are always perfectly elastic. An inelastic collision however, may be partially or perfectly inelastic.

(ii) Visualise a ball dropped from some height on a hard surface, the entire system being in an evacuated space. If the ball rebounds exactly to the same height from where it was dropped, the collision between the ball and the surface (in turn, with the Earth) is elastic. As you know, the ball never reaches the same initial height or a height greater than the initial height. Rebounding to smaller height refers to inelastic collision. Instead of ball, if mud or clay is dropped, it sticks to the surface. This is perfectly inelastic collision.

4.8.3. Coefficient of Restitution *e*:

For collision of two objects, the negative of ratio of relative velocity of separation to relative velocity of approach is defined as the coefficient of restitution *e*.

One dimensional or head-on collision: A collision is said to be head-on if the colliding objects move along the same straight line, before and after the collision. Here, we use u_1, u_2, v_1, v_2 as symbols.

Consider such a head-on collision of two bodies of masses m_1 and m_2 with respective initial velocities u_1 and u_2. As the collision is head on, the colliding masses are along the same line before and after the collision. Hence, vector treatment is not necessary. (**However, velocities must be substituted with proper signs in actual calculation**). Relative velocity of approach is then $u_a = u_2 - u_1$

Let v_1 and v_2 be their respective velocities after the collision. The relative velocity of recede (or separation) is then $\mathrm{v}_s = \mathrm{v}_2 - \mathrm{v}_1$

$$\therefore \text{Coefficient of restitution, } e = -\frac{\mathrm{v}_s}{u_a}$$

$$= \frac{-)\mathrm{v}_2 - \mathrm{v}_1(}{u_2 - u_1} = \frac{\mathrm{v}_1 - \mathrm{v}_2}{u_2 - u_1}$$

$$= \frac{\text{Relative speed of separation}}{\text{Relative speed of approach}} \quad \text{---(4.1)}$$

For a perfectly inelastic collision, the colliding bodies move jointly after the collision, i.e., $\mathrm{v}_2 = \mathrm{v}_1$ or $\mathrm{v}_2 - \mathrm{v}_1 = 0$. Hence, for a perfectly inelastic collision, $e = 0$. In other words, if $e = 0$, the head-on collision is perfectly inelastic collision.

Coefficient of restitution during a head-on, elastic collision:

Consider the collision described above to be elastic. According to the principle of conservation of linear momentum,

Total initial momentum = Total final momentum.

$$\therefore m_1 u_1 + m_2 u_2 = m_1 \mathrm{v}_1 + m_2 \mathrm{v}_2 \quad \text{--- (4.2)}$$

$$\therefore m_1 \left(u_1 - \mathrm{v}_1\right) = m_2 \left(\mathrm{v}_2 - u_2\right) \quad \text{--- (4.3)}$$

As the collision is elastic, total kinetic energy of the system is also conserved.

$$\therefore \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 \mathrm{v}_1^2 + \frac{1}{2} m_2 \mathrm{v}_2^2 \quad \text{--- (4.4)}$$

$$\therefore m_1 \left(u_1^2 - \mathrm{v}_1^2\right) = m_2 \left(\mathrm{v}_2^2 - u_2^2\right)$$

$$\therefore m_1 \left(u_1 + \mathrm{v}_1\right)\left(u_1 - \mathrm{v}_1\right)$$

$$= m_2 \left(u_2 + \mathrm{v}_2\right)\left(\mathrm{v}_2 - u_2\right) \quad \text{--- (4.5)}$$

Dividing Eq. (4.5) by Eq. (4.3), we get

$$u_1 + v_1 = u_2 + v_2 \quad \text{--- (4.6)}$$

$$\therefore u_2 - u_1 = v_1 - v_2$$

For an elastic collision,

$$e = \frac{v_1 - v_2}{u_2 - u_1} = 1 \quad \text{--- (4.7)}$$

Thus, for an elastic collision, coefficient of restitution, $e = 1$. For a perfectly inelastic collision, $e = 0$ (by definition). Thus, for any collision, the coefficient of restitution lies between 1 and 0.

Above expressions (Eq. (4.1) Eq. (4.7)) are general. While substituting the values of u_1, u_2, v_1, v_2, their algebraic values must be used in actual calculation.

For example referring to the

Fig. 4.3 (a), (b) and (c)

Eq (4.1) gives

$$e = -\frac{v_s}{u_a}$$

Here $u_a = u_1 - u_2$ since $u_1 > u_2$

and $v_s = v_1 + v_2$ since the objects go in opposite directions.

$$\therefore e = -\frac{v_1 + v_2}{u_1 - u_2} \quad \text{--- (a)}$$

Using Eq (4.6),

$u_1 + v_1 = u_2 + v_2$

$\therefore$ According to Fig. 4.3,

$u_1 - v_1 = u_2 + v_2$

$\therefore v_1 + v_2 = u_2 - u_1$ --- (b)

By substituting in (a),

$$e = -\frac{(u_2 - u_1)}{(u_1 - u_2)} = 1,$$

which is the case of a perfectly elastic collision.

4.8.4. Expressions for final velocities after a head-on, elastic collision:

From Eq. (4.6), $v_2 = u_1 + v_1 - u_2$

Using this in Eq. (4.2), we get

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 \left(u_1 + v_1 - u_2\right)$$

$$\therefore v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2 \quad \text{--- (4.8)}$$

Subscripts 1 and 2 were arbitrarily chosen. Thus, just interchanging 1 with 2 gives us v_2 as

$$v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2 + \left(\frac{2m_1}{m_1 + m_2}\right) u_1 \quad \text{--- (4.9)}$$

Equation (4.9) can also be obtained by substituting v_1 from Eq. (4.8) in Eq. (4.6).

Particular cases:

(i) If the bodies are of equal masses (or identical), $m_1 = m_2$, Eqs. (4.8) and (4.9) give

$v_1 = u_2$ and $v_2 = u_1$.

Thus, the bodies just exchange their velocities.

(ii) If colliding body is much heavier and the struck body is initially at rest, i.e.,

$$m_1 \gg m_2 \text{ and } u_2 = 0,$$

we can use

$$m_1 \pm m_2 \cong m_1 \text{ and } \frac{m_2}{m_1 + m_2} \cong 0$$

$\therefore v_1 \cong u_1$ and $v_2 \cong 2u_1$, i.e., the massive striking body is practically unaffected and the tiny body which is struck, travels with double the speed of the massive striking body.

(iii) The body which is struck is much heavier than the colliding body and is initially at rest, i.e., $m_1 \ll m_2$ and $u_2 = 0$.

Using similar approximations, we get, $v_1 \cong -u_1$ and $v_2 \cong 0$, i.e., the tiny (lighter) object rebounds with same speed while the massive object is unaffected. This is as good as dropping an elastic object on hard surface of the Earth .

Do you know ?

Are you aware of elasticity of materials? Is there any connection between elasticity of materials and elastic collisions?

Example 4.6: One marble collides head-on with another identical marble at rest. If the collision is partially inelastic, determine the ratio of their final velocities in terms of coefficient of restitution *e*.

Solution: According to conservation of momentum, $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

As $m_1 = m_2$, we get, $u_1 + u_2 = \mathrm{v}_1 + \mathrm{v}_2$

$\therefore$ If $u_2 = 0$, we get, $\mathrm{v}_1 + \mathrm{v}_2 = u_1$ ... (I)

Coefficient of restitution,

$$e = \frac{\mathrm{v}_2 - \mathrm{v}_1}{u_1 - u_2} \quad \therefore \mathrm{v}_2 - \mathrm{v}_1 = eu_1 \text{ ... (II)}$$

Dividing Eq. (I) by Eq. (II),

$$\frac{\mathrm{v}_1 + \mathrm{v}_2}{\mathrm{v}_2 - \mathrm{v}_1} = \frac{1}{e}$$

Using componendo and dividendo, we get,

$$\frac{\mathrm{v}_2}{\mathrm{v}_1} = \frac{1+e}{1-e}$$

4.8.5. Loss in the kinetic energy during a perfectly inelastic head-on collision:

Consider a perfectly inelastic, head on collision of two bodies of masses m_1 and m_2 with respective initial velocities u_1 and u_2. As the collision is perfectly inelastic, they move jointly after the collision, i.e., their final velocity is the same. Let it be v.

According to conservation of linear momentum, $m_1u_1 + m_2u_2 = (m_1 + m_2)\mathrm{v}$

$$\therefore \mathrm{v} = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} \quad \text{--- (4.10)}$$

This is the common velocity after a perfectly inelastic collision

Loss in K.E. = Δ (K.E.)

= Total initial K.E. - Total final K.E.

$$\therefore \Delta(\text{K.E.}) = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2)\mathrm{v}^2$$

$$= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1u_1 + m_2u_2}{m_1 + m_2}\right)^2$$

$$\therefore \Delta(\text{K.E.}) = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}\frac{(m_1u_1 + m_2u_2)^2}{(m_1 + m_2)}$$

On simplifying, we get,

$$\Delta(\text{K.E.}) = \frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)(u_1 - u_2)^2 \quad \text{--- (4.11)}$$

Masses are always positive and $(u_1 - u_2)^2$ is also positive. Hence, there is always a loss in the kinetic energy in a perfectly inelastic collision.

Final velocities and loss in K.E. in an inelastic head-on collision:

If e is the coefficient of restitution, using Eq. (4.2), the expressions for final velocities after an inelastic collision can be derived as

$$\mathrm{v}_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)u_1 + \left(\frac{[1+e]m_2}{m_1 + m_2}\right)u_2$$

$$= \frac{em_2(u_2 - u_1) + m_1u_1 + m_2u_2}{m_1 + m_2}$$

and

$$\mathrm{v}_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)u_2 + \left(\frac{[1+e]m_1}{m_1 + m_2}\right)u_1$$

$$= \frac{em_1(u_1 - u_2) + m_1u_1 + m_2u_2}{m_1 + m_2}$$

Loss in the kinetic energy is given by

$$\Delta(\text{K.E.}) = \frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)(u_1 - u_2)^2(1 - e^2).$$

As $e < 1$, $(1 - e^2)$ is always positive. Thus, there is always a loss of K.E. in an inelastic collision. Also, for a perfectly inelastic collision, $e = 0$. Hence, in this case, the loss is maximum. Using $e = 1$, these equations lead us to an elastic collision and for $e = 0$ they lead us to a perfectly inelastic collision. Verify that they give the same expressions that are derived earlier.

The quantity $\mu = \dfrac{m_1m_2}{m_1 + m_2}$ is the reduced mass of the system.

Impulse or change in momenta of the bodies:

During collision, the linear momentum delivered by first body (particle) to the second body must be equal to change in momentum or *impulse* of the second body, and vice versa.

$$\therefore \text{Impulse, } |J| = |\Delta p_1| = |\Delta p_2|$$

$$= |m_1\mathrm{v}_1 - m_1u_1| = |m_2\mathrm{v}_2 - m_2u_2|$$

On substituting the values of v_1 and v_2 and solving, we get

$$|J| = \left(\frac{m_1m_2}{m_1 + m_2}\right)(1+e)(|u_1 - u_2|)$$

$$= \mu(1+e)u_{\text{relative}}$$

$|u_1 - u_2| = u_{\text{relative}}$ = velocity of approach

4.8.6. Collision in two dimensions, i.e., a nonhead-on collision:

In this case, the direction of at least one initial velocity is NOT along the line of impact. In order to discuss such collisions mathematically, it is convenient to use two mutually perpendicular directions as shown in Fig. 4.4. One of them is the common tangent at the point of impact, along which there is no force (or along this direction, there is no change in momentum). The other direction is perpendicular to this common tangent through the point of impact, in the two-dimensional plane of initial and final velocities. This is called the line of impact. Internal mutual forces exerted during impact, which are responsible for change in the momenta, are acting along this line. From Fig. (4.4), $\vec{u}_1$ and $\vec{u}_2$, initial velocities make angles α_1 and α_2 respectively with the line of impact while $\vec{v}_1$ and $\vec{v}_2$, final velocities make angles β_1 and β_2 respectively with the line of impact.

According to conservation of linear momentum along the line of contact,

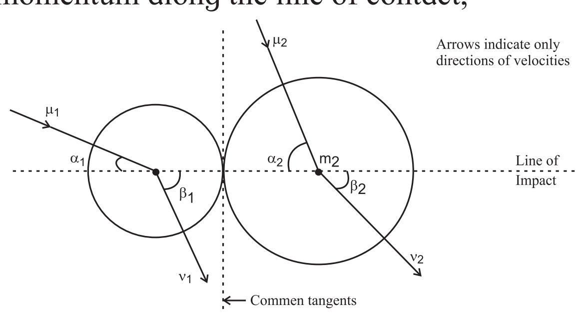


Fig. 4.4: Oblique or non head-on collision.

$$m_1u_1\cos\alpha_1 + m_2u_2\cos\alpha_2$$

$$= m_1\text{v}_1\cos\beta_1 + m_2\text{v}_2\cos\beta_2 \quad \text{--- (4.12)}$$

As there is no force along the common tangent (perpendicular to line of impact),

$$m_1u_1\sin\alpha_1 = m_1\text{v}_1\sin\beta_1 \quad \text{--- (4.13)}$$

and $m_2u_2\sin\alpha_2 = m_2\text{v}_2\sin\beta_2$ --- (4.14)

For coefficient of restitution, along the line of impact,

$$e = -\left(\frac{\text{v}_2\cos\beta_2 - \text{v}_1\cos\beta_1}{u_2\cos\alpha_2 - u_1\cos\alpha_1}\right)$$

$$= \frac{\text{v}_2\cos\beta_2 - \text{v}_1\cos\beta_1}{u_1\cos\alpha_1 - u_2\cos\alpha_2} \quad \text{--- (4.15)}$$

Equations (4.12), (4.13), (4.14) and (4.15) are to be solved for the four unknowns v_1, v_2, β_1 and β_2

Magnitude of the impulse, along the line of impact,

$$|J| = \left(\frac{m_1m_2}{m_1+m_2}\right)(1+e)\left(\left|u_1\cos\alpha_1 - u_2\cos\alpha_2\right|\right)$$

$$= \mu(1+e)u_{\text{relative}}$$

along line of impact.

Loss in the kinetic energy = Δ (K.E.)

$$\frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(u_1\cos\alpha_1 - u_2\cos\alpha_2)^2(1-e^2)$$

$$= \frac{1}{2}\mu\left(u^2_{\text{relative}}\right)(1-e^2)$$

Example 4.7: A shell of mass 3 kg is dropped from some height. After falling freely for 2 seconds, it explodes into two fragments of masses 2 kg and 1 kg. Kinetic energy provided by the explosion is 300 J. Using g = 10 m/s², calculate velocities of the fragments. Justify your answer if you have more than one options.

Solution: $m_1 + m_2 = 3$ kg.

After falling freely for 2 seconds,

$$\text{v} = u + at = 0 + 10(2) = 20\,m\,s^{-1} = u_1 = u_2$$

According to conservation of linear momentum, $m_1u_1 + m_2u_2 = m_1\text{v}_1 + m_2\text{v}_2$

$$\therefore 3\times 20 = 2\text{v}_1 + 1\text{v}_2 \quad \therefore \text{v}_2 = 60 - 2\text{v}_1 \quad \text{--- (I)}$$

K.E. provided = Final K.E. – Initial K.E. $= \frac{1}{2}m_1\text{v}_1^2 + \frac{1}{2}m_2\text{v}_2^2 - \frac{1}{2}(m_1+m_2)u^2$

$$\therefore \frac{1}{2}2\text{v}_1^2 + \frac{1}{2}\text{v}_2^2 - \frac{1}{2}3(20)^2 = 300\,\text{J}$$

$$\text{or } 2\text{v}_1^2 + \text{v}_2^2 = 1800$$

$$\text{or } 2\text{v}_1^2 + (60 - 2\text{v}_1)^2 = 1800 \text{ using Eq. (I)}$$

$$\therefore 3600 - 240\text{v}_1 + 6\text{v}_1^2 = 1800$$

$$\therefore \text{v}_1^2 - 40\text{v}_1 + 300 = 0$$

$$\therefore \text{v}_1 = 30\,m\,s^{-1} \text{ or } 10\,m\,s^{-1} \text{ and}$$

$$\text{v}_2 = 0 \text{ or } 40\,m\,s^{-1}$$

There are two possible answers since the positions of two fragments can be different as explained below.

If $v_1 = 30$ m s^{-1} and $v_2 = 0$, lighter fragment 2 should be above. On the other hand, if $v_1 = 10$ m s^{-1} and $v_2 = 40$ m s^{-1}, lighter fragment 2 should be below, both moving downwards.

Example 4.8: Bullets of mass 40 g each, are fired from a machine gun at a rate of 5 per second towards a firmly fixed hard surface of area 10 cm^2. Each bullet hits normal to the surface at 400 m/s and rebounds in such a way that the coefficient of restitution for the collision between bullet and the surface is 0.75. Calculate average force and average pressure experienced by the surface due to this firing.

Solution: For the collision,

$u_1 = 400\,\text{m s}^{-1}$, $e = 0.75$, $v_1 = ?$

For the firmly fixed hard surface, $u_2 = v_2 = 0$

$$e = 0.75 = \frac{v_1 - v_2}{u_2 - u_1} = \frac{v_1 - 0}{0 - 400} \quad \therefore v_1 = -300 \text{ m/s.}$$

-ve sign indicates that the bullet rebounds in exactly opposite direction.

Change in momentum of each bullet $= m\,(v_1 - u_1)$

Equal and opposite will be the momentum transferred to the surface, per collision.

$\therefore$ Momentum transferred to the surface, per collision

$$p = m\left(u_1 - v_1\right) = 0.04\left(400 - [-300]\right) = 28 \text{ N s}$$

The rate of collision is same as rate of firing.

$\therefore$ Momentum received by the surface per second, $\frac{dp}{dt} = 28 \times 5 = 140$ N

This must be the *average* force experienced by the surface of area $A = 10 \text{ cm}^2 = 10^{-3} \text{ m}^2$

$\therefore$ Average pressure experienced,

$$P = \frac{F}{A} = \frac{140}{10^{-3}} = 1.4 \times 10^5 \text{ N m}^{-2}$$

$\approx$ 1.4 times the atmospheric pressure.

4.9. Impulse of a force:

According to Newton's first law of motion, any unbalanced force changes linear momentum of the system, i.e., basic effect of an unbalanced force is to change the momentum.

According to Newton's second law of motion, $\vec{F} = \frac{d\vec{p}}{dt}$

$$\therefore d\vec{p} = \vec{F}.dt$$

The quantity 'change in momentum' is separately named as *Impulse of the force* $\vec{J}$.

If the force is constant, and is acting for a finite and measurable time, we can write

The change in momentum in time *t*

$$\vec{J} = d\vec{p} = \vec{p}_2 - \vec{p}_1 = \vec{F}.t \qquad \text{---(4.16)}$$

For a given body of mass *m*, it becomes

$$\vec{J} = \vec{p}_2 - \vec{p}_1 = m\left(\vec{v}_2 - \vec{v}_1\right) = \vec{F}.t \qquad \text{---(4.17)}$$

If $\vec{F}$ is not constant but we know how it varies with time, then

$$\vec{J} = \Delta\vec{p} = \int d\vec{p} = \int \vec{F}.dt \qquad \text{--- (4.18)}$$

Always remember:

1) Colliding objects experience forces along the line of impact which changes their momenta. For their system, these forces are internal forces. These forces form an action-reaction pair, which are equal and opposite, and act on different objects.
2) There is no force along the common tangent, i.e., perpendicular to the line of contact.
3) In reality, the impact is followed by emission of sound and heat and occasionally light. Thus, in general, part of mechanical energy- kinetic energy - is lost (i.e., converted into some other non-recoverable forms). However, total energy of the system is conserved.
4) In reality, velocity of separation (relative final velocity) is less than velocity of approach (relative initial velocity) along the line of impact. Thus coefficient of restitution e < 1.
5) Only during elastic collisions (atomic and molecular level only, never possible in real life), the kinetic energy is conserved and the velocity of separation is equal to the velocity of approach or the initial relative velocity is equal to the final relative velocity.

4.9.1. Necessity of defining impulse:

As discussed above, if a force is constant over a given interval of time or if we know how it varies with time, we can calculate the corresponding change in momentum directly by multiplying the force and time.

However, in many cases, an appreciable force acts for an extremely small interval of time (too small to measure the force and the time independently). However, change in the momentum due to this force is noticeable and can be measured. This change is defined as *impulse* of the force.

Real life illustrations: While (i) hitting a ball with a bat, (ii) giving a kick to a foot-ball, (iii) hammering a nail, (iv) bouncing a ball from a hard surface, etc., appreciable amount of force is being exerted. In such cases the time for which these forces act on respective objects is negligibly small, mostly not easily recordable. However, the effect of this force is a recordable change in the momentum of that object. Thus, it is convenient to define the change in momentum itself as a physical quantity.

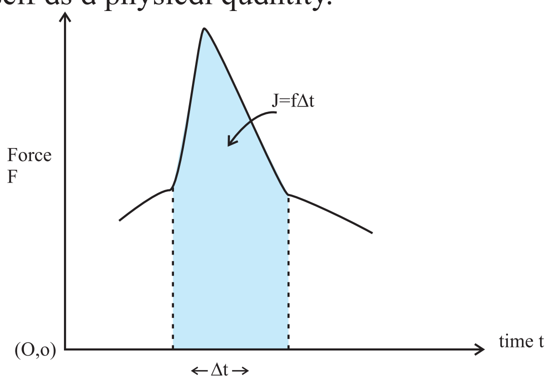


Fig. 4.5: Graphical representation of impulse of a force.

Figure 4.5 shows variation of a force as a function of time e.g., for a collision between bat and ball with the *force axis starting with zero*. The shaded area or the area under the curve gives the product of force against corresponding time (in this case, Δt), hence gives the impulse. For a constant force it is obviously a rectangle. Generally, force is zero before the impact, rises to a maximum and decreases to zero after the impact. For softer tennis ball, the collision time is larger and the maximum force is less. The area under the (F - t) graph is the same. Wicket keeper eases off (by increasing the time of collision) while catching a fast ball. As mentioned earlier, it is absolutely necessary that the force axis must start from zero.

Recall from Chapter 3, analogues concepts using area under a curve are (i) Obtaining displacement in a given time interval as area under the curve for *v-t* graph, with zero origin for velocity axis. (ii) Obtaining work done by a force as the area under the curve for *F-s* graph, with zero origin for force axis.

Example 4.9: Mass of an Oxygen molecule is 5.35×10^{-26} kg and that of a Nitrogen molecule is 4.65×10^{-26} kg. During their Brownian motion (random motion) in air, an Oxygen molecule travelling with a velocity of 400 m/s collides elastically with a nitrogen molecule travelling with a velocity of 500 m/s in the exactly opposite direction. Calculate the impulse received by each of them during collision. Assuming that the collision lasts for 1 ms, how much is the average force experienced by each molecule?

Let

$$m_1 = m_O = 5.35\times10^{-26}\ \text{kg}\ ,$$

$$m_2 = m_N = 4.65\times10^{-26}\,\text{kg}.$$

$$\therefore u_1 = 400\,\text{m s}^{-1}\ and\ u_2 = -500\,\text{m s}^{-1}$$

taking direction of motion of Oxygen molecule as the positive direction.

For an elastic collision,

$$\text{v}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2 \quad \text{and}$$

$$\text{v}_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2 + \left(\frac{2m_1}{m_1 + m_2}\right)u_1$$

$$\therefore \text{v}_1 = -437\,\text{m s}^{-1}\ and\ \ \text{v}_2 = 463\,\text{m s}^{-1}$$

$$\therefore J_O = m_O\left(\text{v}_1 - u_1\right) = -4.478\times10^{-23}\ \text{N s}\ ,$$

$$J_N = m_N\left(\text{v}_2 - u_2\right) = +4.478\times10^{-23}\ \text{N s}$$

As expected, the net impulse or net change in momentum is zero.

$$F_{ON} = \frac{dp_O}{dt} = \frac{J_O}{\Delta t} = \frac{-4.478\times10^{-23}}{10^{-3}}$$

$$= -4.478\times10^{-20}\ \text{N}$$

and $F_{NO} = -F_{ON} = 4.478\times10^{-20}$ N

4.10. Rotational analogue of a force - moment of a force or torque:

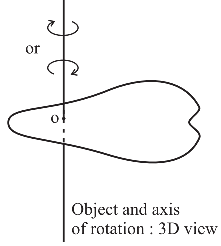
While opening a door fixed to a frame on hinges, we apply the force away from the hinges and perpendicular to the door to open it with ease. In this case we are interested in achieving some angular displacement for the door. If the force is applied near the hinges or nearly parallel to the door, it is very difficult to open the door. Similarly, if the door is heavier (made up of iron instead of wood or plastic), we need to apply proportionally larger force for the same angular displacement.

It shows that rotational ability of a force not only depends upon the mass (greater force for greater mass), but also upon the point of application of the force (the point should be as away as possible from the axis of rotation) and the angle between direction of the force and the line joining the axis of rotation with the point of application (effect is maximum, if this angle is 90^0).

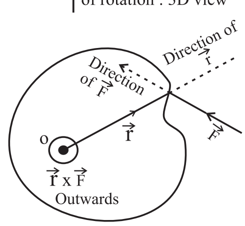
Taking into account all these factors, the quantity **moment of a force** or **torque** is defined as the rotational analogue of a force. As rotation refers to direction (sense of rotation), torque must be a vector quantity. In its mathematical form, torque or moment of a force is given by

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{--- (4.17)}$$

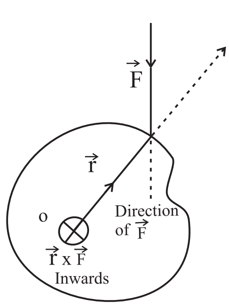
where $\vec{F}$ is the applied force and $\vec{r}$ is the position vector of the point of application of the force from the axis of rotation, as shown in the Figs. 4.6 (b) and 4.6 (c).



Figs. 4.6(a): Illustration of moment of force with object and axis of rotation in 3D view.



Figs. 4.6(b): Top view for moment of force $\vec{F}$ in anticlockwise rotation with $\vec{F}$ and $\vec{r}$ in the plane of paper.



Figs. 4.6(c): Top view of moment of force $\vec{F}$ in clockwise rotation $\vec{F}$ and $\vec{r}$ in the plane of paper.

Figures 4.6(a), 4.6(b) and 4.6(c) illustrate the directions involved. Figure 4.6(a) is a 3D drawing indicating the laminar (plane or two dimensional) object rotating about a (*fixed*) axis of rotation AOB, the axis being perpendicular to the object and passing through it. Figure 4.6(b) indicates the top view of the object when the rotation is in anticlockwise direction and Fig. 4.6(c) shows the view from the top, if rotation is in clockwise direction. (In fact, Figs. 4.6(b) and 4.6(c) are drawn in such a way that the applied force $\vec{F}$ and position vector $\vec{r}$ of the point of application of the force are in the plane of these figures). Direction of the torque is always perpendicular to the plane containing the vectors $\vec{r}$ and $\vec{F}$ and can be obtained from the rule of cross product or by using the right-hand thumb rule. In Fig. 4.6(b), it is perpendicular to the plane of the figure (in this case, perpendicular to the body) and outwards, i.e., coming out of the paper while in the Fig. 4.6(c), it is inwards, i.e., going into the paper.

In order to indicate the directions which are not in the plane of figure, we use a special convention: ⊙ for perpendicular to the plane of figure and outwards and ⊗ for perpendicular to the plane of figure and inwards.

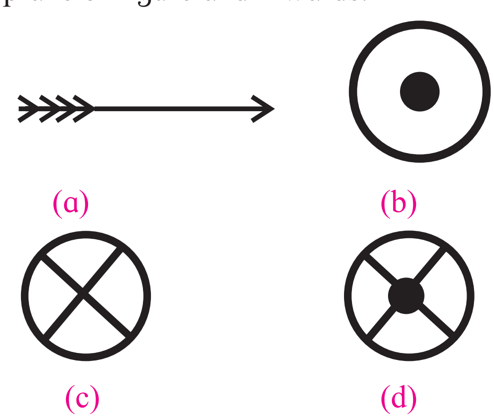


Fig. 4.7: Convention of pictorial representation of vectors as shown in (a) acting in a direction perpendicular to the plane of paper (b) coming out of paper, (c) going in to the paper and (d) perpendicular to the plane of paper.

This convention depends upon a traditional arrow shown in Fig. 4.7 (a). Consider yourself, looking towards the figure from the top. If this arrow approaches you, the tip of the arrow will be prominently seen. Hence circle with a dot in it [Fig. 4.7 (b)] refers to perpendicular and outwards (or towards you). When you are leaving an arrow, i.e., if an arrow is going away from you, the feathers like a cross will be seen. Hence, a circle with a cross [Fig. 4.6 (c)] indicates perpendicular and inwards (or away from you). Circle with cross and dot indicates a line perpendicular to the plane of figure [Fig. 4.6 (d)].

Magnitude of torque, $\tau = r F \sin\theta$ --- (4.18)

where θ is the smaller angle between the *directions* of $\vec{r}$ and $\vec{F}$.

Consequences: (i) If r or F is greater, the torque (hence the rotational effect) is greater. Thus, it is recommended to apply the force away from the hinges.

(ii) If $\theta = 90^0$, $\tau = \tau_{max} = rF$. Thus, the force should be applied along normal direction for easy rotation.

(iii) If $\theta = 0°$ or $180°$, $\tau = \tau_{min} = 0$. Thus, if the force is applied parallel or anti-parallel to $\vec{r}$, there is no rotation.

(iv) Moment of a force depends not only on the magnitude and direction of the force, but also on the point where the force acts with respect to the axis of rotation. Same force can have different torque as per its point of application.

4.11. Couple and its torque:

In the discussion of the torque given above, we had considered rotation of the body about a fixed axis and due to a single force. In real life, quite often we apply two equal and opposite forces acting along different lines of action in order to cause rotation. Common illustrations are turning a bicycle handle, turning the steering wheel, opening a common water tap, opening the lid of a bottle (rotation type), etc. Such a pair of forces consisting of two forces of equal magnitude acting in opposite directions along different lines of action is called a couple. It is used to realise a *purely* rotational motion. Moment of a couple or rotational effect of a couple is also called a *torque*.

It may be noted that in the discussion of rotation of a body about a fixed axis due to a single force, there is a reaction force at the fixed axis. Hence, for rotation one always needs two forces acting in opposite direction along different lines of action.

Torque or Moment of a couple: Figure 4.8 shows a couple consisting of two forces $\vec{F}_1$ *and* $\vec{F}_2$ of equal magnitudes and opposite directions acting along different lines of action separated by a distance r. Corresponding position vectors should now be defined with reference to the lines of action of forces. Position vector of *any* point on the line of action of force $\vec{F}_1$ from the line of action of force $\vec{F}_2$ is $\vec{r}_{12}$. Similarly, the position vector of *any* point on the line of action of force $\vec{F}_2$ from the line of action of force $\vec{F}_1$ is $\vec{r}_{21}$. Torque or moment of the couple is then given mathematically as

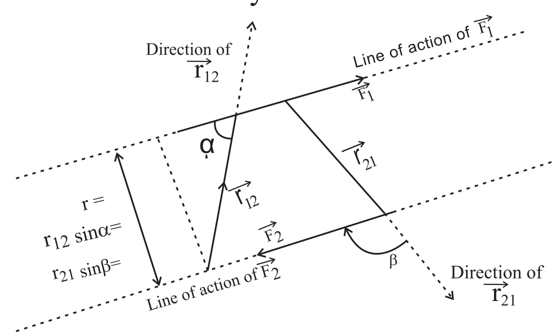


Fig. 4.8: Torque of a couple.

$$\vec{\tau} = \vec{r}_{12} \times \vec{F}_1 = \vec{r}_{21} \times \vec{F}_2 \quad \text{--- (4.19)}$$

From the figure, it is clear that $r_{12} \sin\alpha = r_{21} \sin\beta = r$

If $|\vec{F}_1| = |\vec{F}_2| = F$, the magnitude of torque is given by

$$\tau = r_{12} F_1 \sin\alpha = r_{21} F_2 \sin\beta = r F \quad \text{--- (4.20)}$$

It clearly shows that the torque corresponding to a given couple, i.e., the ***moment of a given couple is constant***, i.e., ***it is independent of the points of application of forces or the position of the axis of rotation***, but depends only upon magnitude of either force and the separation between their lines of action.

The direction of moment of couple can be obtained by using the vector formula of the torque or by using the right-hand thumb rule. For the couple shown in the Fig. 4.8, it is perpendicular to the plane of the figure and inwards. For a given pair of forces, the direction of the torque is fixed.

In many situations the word *couple* is used synonymous to moment of the couple or its torque, i.e., every time we may not say it as torque due to the couple, but say that a *couple* is acting.

	Moment of a force	Moment of a couple
1	$\vec{\tau} = \vec{r} \times \vec{f}$	$\vec{\tau} = \vec{r}_{12} \times \vec{F}_1 = \vec{r}_{21} \times \vec{F}_2$
2	$\vec{\tau}$ depends upon the axis of rotation and the point of application of the force.	$\vec{\tau}$ depends ***only*** upon the two forces, i.e., it is independent of the axis of rotation or the points of application of forces.
3	It can produce translational acceleration also, if the axis of rotation is not fixed or if friction is not enough.	Does not produce any translational acceleration, but produces only rotational or angular acceleration.
4	Its rotational effect can be balanced by a proper single force or by a proper couple.	Its rotational effect can be balanced *only* by another couple of equal and opposite torque.

4.11.1. To prove that the moment of a couple is independent of the axis of rotation:

Figure 4.9 shows a rectangular sheet (any object would do) free to rotate only about a fixed axis of rotation, perpendicular to the plane of figure, as shown. A couple consisting of forces $\vec{F}$ and $-\vec{F}$ is acting on the sheet at different locations.

Here we are considering the torque of a couple to be two torques due to individual forces causing rotation about the axis of rotation. In Fig. 4.9(a), the axis of rotation is between the lines of action of the two forces constituting the couple. Perpendicular distances of the axis of rotation from the forces $\vec{F}$ and $-\vec{F}$ are x and y respectively. Rotation due to the pair of forces *in this case* is anticlockwise (from top view), i.e., directions of individual torques due to the two forces are the same.

$$\therefore \tau = \tau_+ + \tau_- = xF + yF = (x+y)F = rF \quad (4.21)$$

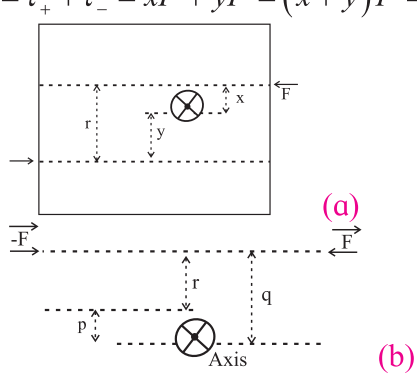


Fig. 4.9: Same couple on same object with fixed axis of rotation at different locations in (a) and (b).

In the Fig. 4.9 (b), lines of action of both the forces are on the same side of the axis of rotation. Thus, *in this case,* the rotation of $+\vec{F}$ is anticlockwise, while that of $-\vec{F}$ is clockwise (from the top view). As a result, their individual torques are oppositely directed. Perpendicular distance of the forces F and $-F$ from the axis of rotation are q and p respectively.

$$\therefore \tau = \tau_+ - \tau_- = qF - pF$$

$$= (q-p)F = rF \quad \text{--- (4.22)}$$

From equations (4.21) and (4.22), it is clear that the torque of a couple is independent of the axis of rotation.

4.12. Mechanical equilibrium:

As a consequence of Newton's second law, the momentum of a system is constant in the absence of an external unbalanced force. This state is called *mechanical equilibrium.*

A particle is said to be in mechanical equilibrium, if no net force is acting upon it. For a system of bodies to be in mechanical equilibrium, the net force acting on *any part* of the system should be zero. In other words, velocity or linear momentum of all parts of the system must be constant or (zero) for the system to be in mechanical equilibrium. Also, there is no acceleration in any part of the system.

Mathematically, $\sum \vec{F} = 0$, for any part of the system for mechanical equilibrium.

4.12.1 Stable, unstable and neutral equilibrium:

Figures 4.10 (a), (b) and (c) show a ball at rest in three situations under the action of balanced forces. In all these cases, it is under equilibrium. However, potential energy-wise, the three cases differ.

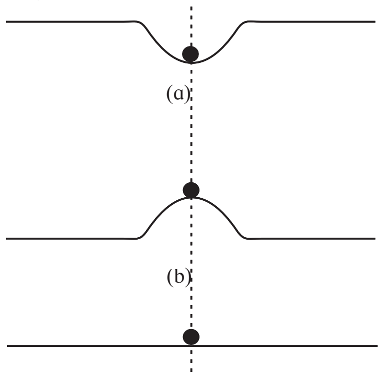


Fig. 4.10: states of mechanical equilibrium (a) stable, (b) unstable and (c) neutral.

Stable equilibrium: In Fig. 4.9(a), the ball is most stable and is said to be in *stable* equilibrium. If it is disturbed slightly from its equilibrium position and released, it tends to recover its position. In this case, potential energy of the system is at its local minimum.

Unstable equilibrium: In Fig. 4.9(b), the ball is said to be in *unstable* equilibrium. If it is slightly disturbed from its equilibrium position, it moves farther from that position. This happens because initially, potential energy of the system is at its local maximum. If disturbed, it tries to achieve the configuration of minimum potential energy.

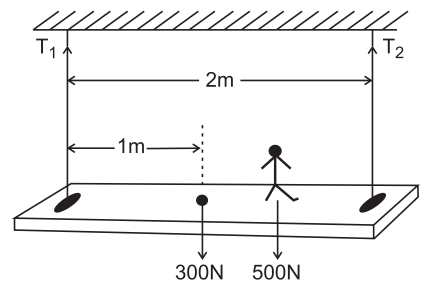
If potential energy function is known for the system, mathematically, the three equilibria can be explained with the help of derivatives of that function. At any equilibrium position, the first derivative of the potential energy function is zero $\left(\frac{dU}{dx}=0\right)$.

The sign of the second derivative $\left(\frac{d^2U}{dx^2}\right)$ decides the type of equilibrium. It is positive

at stable equilibrium (or vice versa), negative at unstable equilibrium and zero (or does not exist) at neutral equilibrium configuration.

Neutral equilibrium: In Fig. 4.9(c), potential energy of the system is constant over a plane and remains same at any position. Thus, even if the ball is disturbed, it still remains in equilibrium at practically any position. This is described as *neutral* equilibrium.

Example 4.10: A uniform wooden plank of mass 30 kg is supported symmetrically by two light identical cables; each can sustain a tension up to 500 N. After tying, the cables are exactly vertical and are separated by 2 m. A boy of mass 50 kg, standing at the centre of the plank, is interested in walking on the plank. How far can he walk? (g = 10 ms^{-2})



Solution: Let T_1 and T_2 be the tensions along the cables, both acting vertically upwards.

Weight of the plank 300 N is acting vertically downwards through the centre, 1 m from either cable. Weight of the boy, 500 N is vertically downwards at the point where he is standing.

$\therefore T_1 + T_2 = 300 + 500 = 800\ \text{N}$

Suppose that the boy is able to walk x m towards the right. Obviously, the tension in the right side cable goes on increasing as he walks towards the cable.

Moments of 300 N and 500 N forces about left end A are clockwise, while that of T_2 is anticlockwise.

As the cable can sustain 500 N, $(T_2)_{max}$ = 500 N

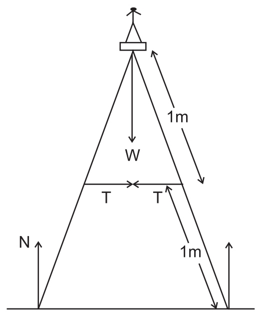
Thus, for the equilibrium about A, we can write,

$300\times1+500\times(1+x)=500\times2 \quad \therefore x=0.4\ \text{m}$

Thus, the boy can walk up to 40 cm on either side of the centre.

Example 4.11: A ladder of negligible mass having a cross bar is resting on a frictionless horizontal floor with angle between its legs to be 40^0. Each leg is 1 m long. Calculate the force experienced by the cross bar when a person of mass 50 kg is standing on the ladder. ($g = 10\ m\ s^{-2}$)

Solution: Tension T along the cross bar is horizontal. Let L be the length of each leg, which is 1 m.



As there is no friction, there is no horizontal reaction at the floor. Reaction N given by the floor at the base of the ladder will then be only vertical. Thus, along the vertical, two such reactions balance weight $W = mg$ of the person.

$$\therefore N = \frac{mg}{2} = 250\ \text{N}$$

At the left leg, about the upper end, the torque due to N is clockwise and that due to the tension T is anticlockwise. For equilibrium, these two torques should have same magnitude.

$$\therefore N.L\sin 20^0 = T.\left(\frac{L}{2}\right)\cos 20^0$$

$$\therefore T = 2N\tan 20^0 = 2\times 250\times 0.364 = 182\ \text{N}$$

4.13. Centre of mass:

As discussed earlier, Newton's laws of motion and many other laws are applicable for point masses only. However, in real life, we always come across finite objects (objects of measurable sizes). Concept of centre of mass (c.m.) helps us in considering these objects to be point objects at a particular location, thereby allowing us to apply Newton's laws of motion.

4.13.1. Mathematical understanding of centre of mass:

(i) System of n particles: Consider a system of n particles of masses $m_1, m_2 \ldots m_n$.

$$\therefore \sum_1^n m_i = M \text{ the total mass.}$$

Let $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n$ be their respective position vectors from a given origin O (Fig. 4.11) .

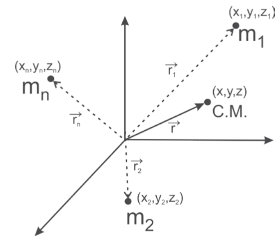


Fig. 4.11: Centre of mass for n particles.

Position vector $\vec{r}$ of their centre of mass from the same origin is then given by

$$\vec{r} = \frac{\sum_1^n m_i\vec{r}_i}{\sum_1^n m_i} = \frac{\sum_1^n m_i\vec{r}_i}{M}$$

If the origin itself is at the centre of mass,

$\vec{r} = 0 \ \therefore \sum_1^n m_i\vec{r}_i = 0$, then

$\sum_1^n m_i\vec{r}_i$ gives the moment of masses (similar to moment of force) about the centre of mass.

Thus, centre of mass is a point about which the summation of moments of masses in the system is zero.

If $x_1, x_2, \ldots x_n$ are the respective x-coordinates of $r_1, r_2, \ldots r_n$, the x-coordinate of the centre of mass is given by

$$x = \frac{\sum_1^n m_i x_i}{\sum_1^n m_i} = \frac{\sum_1^n m_i x_i}{M}$$

Similarly, y and z-coordinates of the centre of mass are respectively given by

$$y = \frac{\sum_1^n m_i y_i}{\sum_1^n m_i} = \frac{\sum_1^n m_i y_i}{M}$$

and

$$z = \frac{\sum_1^n m_i z_i}{\sum_1^n m_i} = \frac{\sum_1^n m_i z_i}{M}$$

(i) Continuous mass distribution: For a continuous mass distribution ***with uniform density***, we need to use integration instead of summation. In this case, the position vector of the centre of mass is given by

$$\vec{r} = \frac{\int \vec{r}\,dm}{\int dm} = \frac{\int \vec{r}\,dm}{M},$$

where $\int dm = M$ is the total mass of the object. Then the Cartesian coordinates of c.m. are

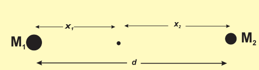
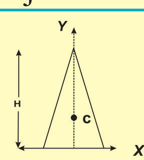
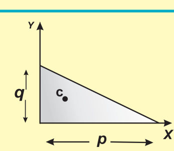
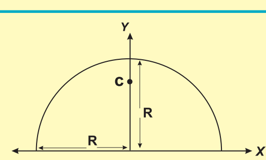
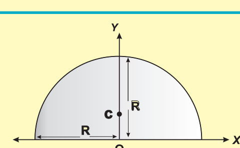
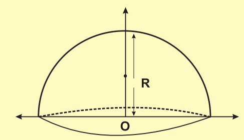
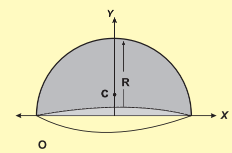
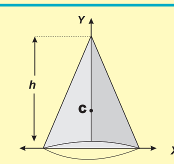
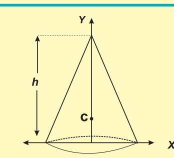
$$x = \frac{\int x\, dm}{\int dm} = \frac{\int x\, dm}{M}$$

$$y = \frac{\int y\, dm}{\int dm} = \frac{\int y\, dm}{M}$$

$$z = \frac{\int z\, dm}{\int dm} = \frac{\int z\, dm}{M}$$

Using the expressions given above, the centres of mass of uniform symmetric objects can be obtained. Some of these are listed in the Table 4.1 given below:

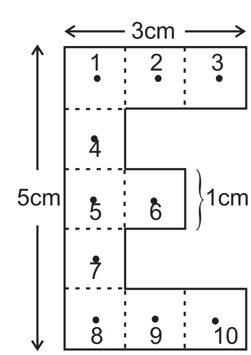
Table 4.1: Coordinate of the centre of mass (c.m.) for some symmetrical objects

Coordinates of c.m.	**Uniform Symmetric Objects**
System of two point masses: c.m. divides the distance in inverse proportion of the masses	
Any geometrically symmetric object of uniform density.	Centre of mass at a geometrical centre of the object
Isosceles triangular plate $x_c = 0, y_c = \frac{H}{3}$	
Right angled triangular plate $x_c = \frac{p}{3},\ y_c = \frac{q}{3}$	
Thin semicircular ring of radius R $x_c = 0,\ y_c = \frac{2R}{\pi}$	
Thin semicircular disc of radius R $x_c = 0,\ y_c = \frac{4R}{3\pi}$	
Hemispherical shell of radius R $x_c = 0,\ y_c = \frac{R}{2}$	
Solid hemisphere of radius R $x_c = 0,\ y_c = \frac{3R}{8}$	
Hollow right circular cone of height h $x_c = 0,\ y_c = \frac{h}{3}$	
Solid right circular cone of height h $x_c = 0,\ y_c = \frac{h}{4}$	

Example 4.12: A letter 'E' is prepared from a uniform cardboard with shape and dimensions as shown in the figure. Locate its centre of mass.

Solution: As the sheet is uniform, each square can be taken to be equivalent to mass m concentrated at its respective centre. These masses will then be at the points labelled with numbers 1 to 10, as shown in figure. Let us select the origin to be at the left central mass m_5, as shown and all the co-ordinates to be in cm.

By symmetry, the centre of mass of m_1, m_2 and m_3 will be at $m_2(1, 2)$ having effective mass $3m$. Similarly, effective mass $3m$ due to m_8, m_9 and m_{10} will be at $m_9(1, -2)$. Again, by symmetry, the centre of mass of these two ($3m$ each) will have co-ordinates (1, 0). Mass m_6 is also having co-ordinates (1, 0). Thus, the effective mass at (1, 0) is $7m$.



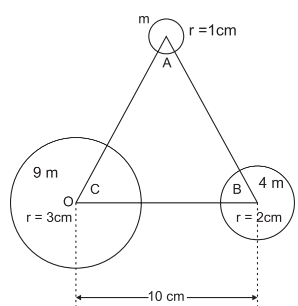
Using symmetry for m_4, m_5 and m_7, there will be effective mass $3m$ at the origin (0, 0).

Thus, effectively, $3m$ and $7m$ are separated by 1 cm along x-direction. y-coordinate is not required.

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{3\times 0 + 7\times 1}{3+7} = 0.7\,\text{cm}$$

Alternately, for two point masses, the centre of mass divides the distance between them in the inverse ratio of their masses. Hence, 1 cm is divided in the ratio 7:3. $\therefore x_c = \frac{7}{7+3}\times 1 = 0.7\,\text{cm}$ from $3m$, i.e., from the origin at m_c

Example 4.13: Three thin walled uniform hollow spheres of radii 1cm, 2 cm and 3 cm are so located that their centres are on the three vertices of an equilateral triangle ABC having each side 10 cm. Determine centre of mass of the system.



Solution: Mass of a thin walled uniform hollow sphere is proportional to its surface area, (as density is constant) hence proportional to r^2. Thus, if mass of the sphere at A is $m_A = m$, then $m_B = 4m$ and $m_C = 9m$. By symmetry of the spherical surface, their centres of mass are at their respective centres, i.e., at A, B and C.

Let us choose the origin to be at C, where the largest mass $9m$ is located and the point B with mass $4m$ on the positive x-axis. With this, the co-ordinates of C are (0, 0) and that of B are (10, 0). (Locating the origin at the larger mass here save our efforts of calculations like multiplications with larger numbers). If A of mass m is taken in the first quadrant, its co-ordinates will be $\left[5, \frac{10\sqrt{3}}{2}\right]$

$$x_c = \frac{m_A x_A + m_B x_B + m_C x_C}{m_A + m_B + m_C}$$

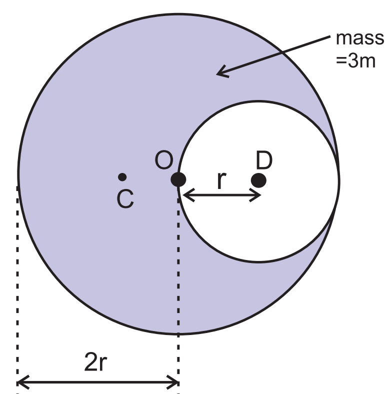
$$= \frac{m\times 5 + 4m\times 10 + 9m\times 0}{m + 4m + 9m} = \frac{45}{14}\,\text{cm}$$

$$y_c = \frac{m_A y_A + m_B y_B + m_C y_C}{m_A + m_B + m_C}$$

$$= \frac{m\times\frac{10\sqrt{3}}{2} + 4m\times 0 + 9m\times 0}{m + 4m + 9m} = \frac{10\sqrt{3}}{28}\,\text{cm}$$

Example 4.14: A hole of radius r is cut from a uniform disc of radius $2r$. Centre of the hole is at a distance r from centre of the disc. Locate centre of mass of the remaining part of the disc.

Solution: Method I: (Using entire disc): Before cutting the hole, c.m. of the full disc was at its centre. Let this be our origin O. Centre of mass of the cut portion is at its centre D. Thus, it is at a distance $x_1 = r$ form the origin. Let C be the centre of mass of the remaining disc. Obviously, it should be on the extension of the line DO. Let it be at a distance $x_2 = x$ from the origin. As the disc is uniform, mass of any of its part is proportional to the area of that part.



Thus, if m is the mass of the cut disc, mass of the entire disc must be $4m$ and mass of the remaining disc will be $3m$.

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

As centre of mass of the full disc is at the origin, we can write,

$$0 = \frac{m\times r + 3m\times(x)}{m + 3m} \quad \therefore x = \frac{-r}{3}$$

Method II: (Using negative mass): Let $\vec{R}$ be the position vector of the centre of mass of the uniform disc of mass M. Mass m is with centre of mass at position vector $\vec{r}$ from the centre of the disc. Position vector of the centre of mass of the remaining disc is then given by

$\vec{r}_c = \dfrac{M\vec{R} - m\vec{r}}{M - m}$ ….. (as if there is a negative mass, i.e., $m_2 = - m$)

With our description, $M = 4m$, $m = m$, $R = 0$ and r = r $\therefore r_c = \dfrac{-mr}{3m} = \dfrac{-r}{3}$ ... Same as method I.

4.13.2. Velocity of centre of mass:

Let $v_1, v_2, \ldots v_n$ be the velocities of a system of point masses $m_1, m_2, \ldots m_n$. Velocity of the centre of mass of the system is given by

$$\vec{v}_{cm} = \frac{\sum_1^n m_i \vec{v}_i}{\sum_1^n m_i} = \frac{\sum_1^n m_i \vec{v}_i}{M}$$

$$= \frac{\text{resultant linear momentum}}{\text{total mass}}$$

= weighted average of momenta

x, y and z components of $\vec{v}$ can be obtained similarly.

For continuous distribution, $\vec{v}_{cm} = \dfrac{\int \vec{v}\, dm}{M}$

4.13.3. Acceleration of the centre of mass:

Let $a_1, a_2, \ldots a_n$ be the accelerations of a system of point masses $m_1, m_2, \ldots m_n$. Acceleration of the centre of mass of the system is given by

$$\vec{a}_{cm} = \frac{\sum_1^n m_i \vec{a}_i}{\sum_1^n m_i} = \frac{\sum_1^n m_i \vec{a}_i}{M}$$

$$= \frac{\text{resultant force}}{\text{total mass}}$$

= weighted average of forces

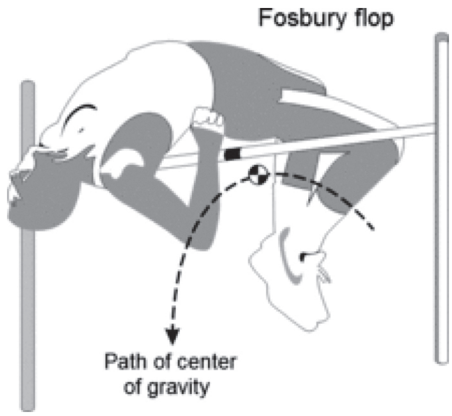
x, y and z components of $\vec{a}$ can be obtained similarly.

For continuous distribution, $\vec{a}_{cm} = \dfrac{\int \vec{a}\, dm}{M}$

4.13.4. Characteristics of centre of mass:

1. Centre of mass is a hypothetical point at which entire mass of the body can be assumed to be concentrated.
2. Centre of mass is a location, and not a physical quantity.
3. Centre of mass is particle equivalent of a given object for applying laws of motion.
4. Centre of mass is the point at which, if a force is applied, it causes only linear acceleration and not angular acceleration.
5. Centre of mass is located at the centroid, for a rigid body ***of uniform density.***
6. Centre of mass is located at the geometrical centre, for a symmetric rigid body ***of uniform density.***
7. Location of centre of mass can be changed ***only*** by an external unbalanced force.
8. Internal forces (like during collision or explosion) ***never*** change the location of centre of mass.
9. Position of the centre of mass depends only upon the distribution of mass, however, to describe its location we may use a coordinate system with a *suitable* origin. In statistical terms the centre of mass is decided by the *weighted* average of individual masses. This is obtained by giving proper mass *weightage* to the distance. This should be clear from the mathematical expression for the location of the centre of mass.
10. For a system of particles, the centre of mass *need not* coincide with any of the particles.
11. While balancing an object on a pivot, the line of action of *weight* must pass through the centre of mass and the pivot. Quite often, this is an unstable equilibrium.
12. Centre of mass of a system of only two particles divides the distance between the particles in an inverse ratio of their masses, i.e., it is closer to the heavier mass.
13. Centre of mass is a point about which the summation of moments of masses in the system is zero.
14. If there is an axial symmetry for a given object, the centre of mass lies on the axis of symmetry.
15. If there are multiple axes of symmetry for a given object, the centre of mass is at their point of intersection.
16. Centre of mass need not be within the

body (See the photograph given below: Picture 4.1). Other examples are a ring, a horse shoe, etc.



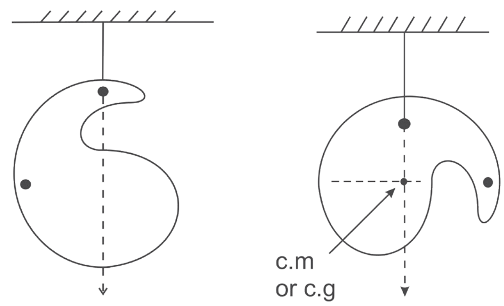
Picture 4.1: Courtesy Wikipedia: Estimated center of mass/gravity of a high jumper doing a Fosbury Flop. Note that it is below the bar in this position. This is possible because our head and legs are much heavier than the fleshy part. Increase in the gravitational potential energy of the high jumper depends upon this point.

4.14. Centre of gravity

Centre of gravity (c.g.) of a body is the point around which the resultant torque due to force of gravity on the body is zero. Analogous to centre of mass, it is the weighted average of the gravitational forces (weights) on individual particles.

For uniform gravitational field (in simple words, if g is constant), c.g. always coincides with the c.m. Obviously it is true for all the objects *on the Earth* in our daily life. Thus, in common usage, the terms c.g. and c.m. are used for same purpose. This property can be used to determine the c.g. (or c.m.) of a laminar (laminar means like a leaf – two dimensional) object.

In Fig. 4.12, a laminar object is suspended from a rigid support at two orientations. Lines are to be drawn on the object parallel to the plumb line shown. Plumb line is always vertical, i.e., parallel to the line of action of gravitational force. Intersection of the lines drawn is then the point through which line of action of the gravitational force passes for any orientation. Thus, it gives the location of the c.g. or c.m.



Centre of mass is a fixed property for a given rigid body in spite of any orientation. The centre of gravity may depend upon non-uniformity of the gravitational field, in turn, will depend upon the orientation. For objects on the Earth , this will be possible only if the size of an object is comparable to that of the Earth (size at least few thousand km). In such cases, the c.g. will be slightly lower than the c.m. as on the lower side of an object the gravitational field is stronger. Of course, we shall not come across such an object.

Exercises

1. Choose the correct answer.

i) Consider following pair of forces of equal magnitude and opposite directions:

(P) Gravitational forces exerted on each other by two point masses separated by a distance.

(Q) Couple of forces used to rotate a water tap.

(R) Gravitational force and normal force experienced by an object kept on a table.

For which of these pair/pairs the two forces do NOT cancel each other's translational effect?

(A) Only P (B) Only P and Q

(C) Only R (D) Only Q and R

ii) Consider following forces: (w) Force due to tension along a string, (x) Normal force given by a surface, (y) Force due to air resistance and (z) Buoyant force or upthrust given by a fluid.

Which of these are electromagnetic forces?

(A) Only w, y and z

(B) Only w, x and y
(C) Only x, y and z
(D) All four.

iii) At a given instant three point masses m, $2m$ and $3m$ are equidistant from each other. Consider only the gravitational forces between them. Select correct statement/s for this instance only:
(A) Mass m experiences maximum force.
(B) Mass $2m$ experiences maximum force.
(C) Mass $3m$ experiences maximum force.
(D) All masses experience force of same magnitude.

iv) The rough surface of a horizontal table offers a definite *maximum* opposing force to initiate the motion of a block along the table, which is proportional to the resultant normal force given by the table. Forces F_1 and F_2 act at the same angle θ with the horizontal and both are just initiating the sliding motion of the block along the table. Force F_1 is a pulling force while the force F_2 is a pushing force. $F_2 > F_1$, because
(A) Component of F_2 adds up to weight to increase the normal reaction.
(B) Component of F_1 adds up to weight to increase the normal reaction.
(C) Component of F_2 adds up to the opposing force.
(D) Component of F_1 adds up to the opposing force.

v. A mass $2m$ moving with some speed is *directly* approaching another mass m moving with double speed. After some time, they collide with coefficient of restitution 0.5. Ratio of their respective speeds after collision is
(A) 2/3 (B) 3/2
(C) 2 (D) ½

vi. A uniform rod of mass $2m$ is held horizontal by two sturdy, practically inextensible vertical strings tied at its ends. A boy of mass $3m$ hangs himself at one third length of the rod. Ratio of the tension in the string close to the boy to that in the other string is
(A) 2 (B) 1.5
(C) 4/3 (D) 5/3

vii. Select WRONG statement about centre of mass:
(A) Centre of mass of a 'C' shaped uniform rod can never be a point on that rod.
(B) If the line of action of a force passes through the centre of mass, the moment of that force is zero.
(C) Centre of mass of our Earth is not at its geometrical centre.
(D) While balancing an object on a pivot, the line of action of the gravitational force of the earth passes through the centre of mass of the object.

viii. For which of the following objects will the centre of mass NOT be at their geometrical centre?
(I) An egg
(II) a cylindrical box full of rice
(III) a cubical box containing assorted sweets
(A) Only (I)
(B) Only (I) and (II)
(C) Only (III)
(D) All, (I), (II) and (III).

2. Answer the following questions.

i) In the following table, every entry on the left column can match with any number of entries on the right side. Pick up ***all*** those and write respectively against A, B, C and D.

	Name of the force		Type of the force
A	Force due to tension in a string	P	EM force
B	Normal force	Q	Reaction force
C	Frictional force	R	Conservative force
D	Resistive force offered by air or water for objects moving through it.	S	Non-conservative force

ii) In real life objects, never travel with uniform velocity, even on a horizontal surface, unless *something* is done? Why is it so? What is to be done?

iii) For the study of any kind of motion, we never use Newton's first law of motion directly. Why should it be studied?

iv) Are there any situations in which we cannot apply Newton's laws of motion? Is there any alternative for it?

v) You are inside a closed capsule from where you are not able to see anything about the outside world. Suddenly you feel that you are pushed towards your right. Can you explain the possible cause (s)? Is it a feeling or a reality? Give at least one more situation like this.

vi) Among the four fundamental forces, only one force governs your daily life almost entirely. Justify the statement by stating that force.

vii) Find the odd man out: (i) Force responsible for a string to become taut on stretching (ii) Weight of an object (iii) The force due to which we can hold an object in hand.

viii) You are sitting next to your friend on ground. Is there any gravitational force of attraction between you two? If so, why are you not coming together naturally? Is any force other than the gravitational force of the earth coming in picture?

ix) Distinguish between: (A) Real and pseudo forces, (B) Conservative and non-conservative forces, (C) Contact and non-contact forces, (C) Inertial and non-inertial frames of reference.

x) State the formula for calculating work done by a force. Are there any conditions or limitations in using it directly? If so, state those clearly. Is there any mathematical way out for it? Explain.

xi) Justify the statement, "Work and energy are the two sides of a coin".

xii) From the terrace of a building of height H, you dropped a ball of mass m. It reached the ground with speed v. Is the relation $mgH = \frac{1}{2}mv^2$ applicable exactly? If not, how can you account for the difference? Will the ball bounce to the same height from where it was dropped?

xiii) State the law of conservation of linear momentum. It is a consequence of which law? Given an example from our daily life for conservation of momentum. Does it hold good during burst of a cracker?

xiv) Define coefficient of restitution and obtain its value for an elastic collision and a perfectly inelastic collision.

xv) Discuss the following as special cases of elastic collisions and obtain their exact or approximate final velocities in terms of their initial velocities.

(i) Colliding bodies are identical.

(ii) A veru heavy object collides on a lighter object, initially at rest.

(iii) A very light object collides on a comparatively much massive object, initially at rest.

xvi) A bullet of mass m_1 travelling with a velocity u strikes a stationary wooden block of mass m_2 and gets embedded into it. Determine the expression for loss in the kinetic energy of the system. Is this violating the principle of conservation of energy? If not, how can you account for this loss?

xvii) One of the effects of a force is to change the momentum. Define the quantity related to this and explain it for a variable force. Usually when do we define it instead of using the force?

xviii) While rotating an object or while opening a door or a water tap we apply a force or forces. Under which conditions is this process easy for us? Why? Define the vector quantity concerned. How does it differ for a single force and for two opposite forces with different lines of action?

xix) Why is the moment of a couple independent of the axis of rotation even if the axis is fixed?

xx) Explain balancing or mechanical equilibrium. Linear velocity of a rotating fan as a whole is generally zero. Is it in mechanical equilibrium? Justify your answer.

xxi) Why do we need to know the centre of mass of an object? For which objects, its position may differ from that of the centre of gravity?

Use g* = *10* m s^{-2}, *unless, otherwise stated.

3. Solve the following problems.

i) A truck of mass 5 ton is travelling on a horizontal road with 36 km hr $^{-1}$ stops on traveling 1 km after its engine fails suddenly. What fraction of its weight is the frictional force exerted by the road?

If we assume that the story repeats for a car of mass 1 ton i.e., can moving with same speed stops in similar distance same how much will the fraction be?

[Ans: $\frac{1}{200}$ in the both]

ii) A lighter object A and a heavier object B are initially at rest. Both are imparted the same linear momentum. Which will start with greater kinetic energy: A or B or both will start with the same energy?

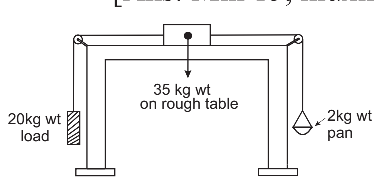
[Ans: A]

iii) As I was standing on a weighing machine inside a lift it recorded 50 kg wt. Suddenly for few seconds it recorded 45 kg wt. What must have happened during that time? Explain with complete numerical analysis.

[Ans: Lift must be coming down with acceleration $\frac{g}{10} = 1\text{m s}^{-2}$]

iv) Figure below shows a block of mass 35 kg resting on a table. The table is so rough that it offers a self adjusting resistive force 10% of the weight of the block for its sliding motion along the table. A 20 kg wt load is attached to the block and is passed over a pulley to hang freely on the left side. On the right side there is a 2 kg wt pan attached to the block and hung freely. Weights of 1 kg wt each, can be added to the pan. Minimum how many and maximum how many such weights can be added into the pan so that the block does not slide along the table?

[Ans: Min 15, maximum 21].



v) Power is rate of doing work or the rate at which energy is supplied to the system. A constant force F is applied to a body of mass m. Power delivered by the force at time t from the start is proportional to

(a) t (b) t^2

(c) $\sqrt{t}$ (d) t^0

Derive the expression for power in terms of F, m and t.

[Ans: $p = \frac{F^2 t}{m}, \therefore p \alpha t$]

vi) 40000 litre of oil of density 0.9 g cc is pumped from an oil tanker ship into a storage tank at 10 m higher level than the ship in half an hour. What should be the power of the pump?

[Ans: 2 kW]

vii) Ten identical masses (m each) are connected one below the other with 10 strings. Holding the topmost string, the system is accelerated upwards with acceleration g/2. What is the tension in the 6th string from the top (Topmost string being the first string)?

[Ans: 6 mg]

viii) Two galaxies of masses 9 billion solar mass and 4 billion solar mass are 5 million light years apart. If, the Sun has to cross the line joining them, without being attracted by either of them, through what point it should pass?

[Ans: 3 million light years from the 9 billion solar mass]

ix) While decreasing linearly from 5 N to 3 N, a force displaces an object from 3 m to 5 m. Calculate the work done by this force during this displacement.

[Ans: 8 N]

x) Variation of a force in a certain region is given by $F = 6x^2 - 4x - 8$. It displaces an object from $x = 1$ m to $x = 2$ m in this region. Calculate the amount of work done. [Ans: Zero]

xi) A ball of mass 100 g dropped on the ground from 5 m bounces repeatedly. During every bounce 64% of the potential energy is converted into kinetic energy. Calculate the following:

(a) Coefficient of restitution.

(b) Speed with which the ball comes up from the ground after third bounce.

(c) Impulse given by the ball to the ground during this bounce.

(d) Average force exerted by the ground

if this impact lasts for 250 ms.

(e) Average pressure exerted by the ball on the ground during this impact if contact area of the ball is 0.5 cm^2.

[Ans: 0.8, 5.12 m/s, 1.152N s, 4.608 N, 9.216×10^4 N/m^2]

xii) A spring ball of mass 0.5 kg is dropped from some height. On falling freely for 10 s, it explodes into two fragments of mass ratio 1:2. The lighter fragment continues to travel downwards with speed of 60 m/s. Calculate the kinetic energy supplied during explosion.

[Ans: 200 J]

xiii) A marble of mass $2m$ travelling at 6 cm/s is directly followed by another marble of mass m with double speed. After collision, the heavier one travels with the average initial speed of the two. Calculate the coefficient of restitution.

[Ans: 0.5]

xiv) A, 2 m long wooden plank of mass 20 kg is pivoted (supported from below) at 0.5 m from either end. A person of mass 40 kg starts walking from one of these pivots to the farther end. How far can the person walk before the plank topples?

[Ans: 1.25 m]

xv) A 2 m long ladder of mass 10 kg is kept against a wall such that its base is 1.2 m away from the wall. The wall is smooth but the ground is rough. Roughness of the ground is such that it offers a maximum horizontal resistive force (for sliding motion) half that of normal reaction at the point of contact. A monkey of mass 20 kg starts climbing the ladder. How far can it climb ***along*** the ladder? How much is the horizontal reaction at the wall?

[Ans: 1.5 m, 15 N]

xvi) Four uniform solid cubes of edges 10 cm, 20 cm, 30 cm and 40 cm are kept on the ground, touching each other in order. Locate centre of mass of their system.

[Ans: 65 cm, 17.7 cm]

xvii) A uniform solid sphere of radius R has a hole of radius $R/2$ drilled inside it. One end of the hole is at the centre of the sphere while the other is at the boundary. Locate centre of mass of the remaining sphere.

[Ans: -R/14]

xviii) In the following table, every item on the left side can match with any number of items on the right hand side. Select all those.

***

Types of collision	Illustrations
(a) Elastic collision (b) Inelastic collision (c) Perfectly inelastic collision (d) Head on collision	(i) A ball hit by a bat. (ii) Molecular collisions responsible for pressure exerted by a gas. (iii) A stationary marble A is hit by marble B and the marble B comes to rest. (iv) A blob of clay dropped on the ground sticks to the ground. (v) Out of anger, giving a kick to a wall. (vi) A striker hits the boundary of a carrom board in a direction perpendicular to the boundary and rebounds.

5. Gravitation

Can you recall?

1. When released from certain height why do objects tend to fall vertically downwards?
2. What is the shape of the orbits of planets? 3. What are Kepler's laws?

5.1 Introduction:

All material objects have a natural tendency to get attracted towards the Earth. In many natural phenomena like coconut falling from trees, raindrops falling from the clouds, etc., the same tendency is observed. All bodies are attracted towards the Earth with constant acceleration. This fact was recognized by Italian physicist Galileo. He is said to have demonstrated it by releasing two balls of different masses from top of the leaning tower of Pisa which reached the ground at the same time.

Indian astronomer and mathematician Aryabhatta (476-550 A.D.) studied the motion of the moon, Earth and other planets in the 5th century A.D. In his book 'Aryabhatiya', he concluded that the Earth revolves about its own axis and it moves in a circular orbit around the Sun. Also the moon revolves in a circular orbit around the Earth. Almost a thousand years after Aryabhatta, Tycho Brahe (1546-1601) and Johannes Kepler (1571-1630) studied planetary motion through careful observations. Kepler analysed the huge data meticulously recorded by Tycho Brahe and established three laws of planetary motion. He showed that the motion of planets follow these laws. The reason why planets obey these laws was provided by Newton. He explained that gravitation is the phenomenon responsible for keeping planets in their orbits around the Sun. The moon also revolves around the Earth due to gravitation. Gravitation compels dispersed matter to coalesce, hence the existence of the Earth, the Sun and all material macroscopic objects in the universe.

Every massive object in the universe experiences gravitational force. It is the force of mutual attraction between any two objects by virtue of their masses. It is always an attractive force with infinite range. It does not depend upon intervening medium. It is much weaker than other fundamental forces. Gravitational force is 10^{-39} times weaker than strong nuclear force.

5.2 Kepler's Laws:

Kepler's laws of planetary motion describe the orbits of the planets around the Sun. He published first two laws in 1609 and the third law in 1619. These laws are the result of the analysis of the data collected by Tycho Brahe through years of observations of the planetary motion.

Do you know ?

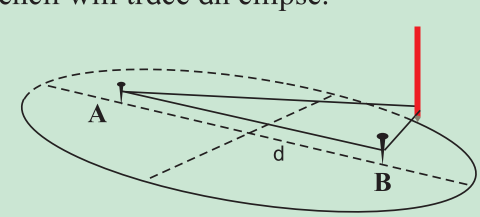
Drawing an ellipse

An ellipse is the locus of the points in a plane such that the sum of their distances from two fixed points, called the foci, is constant.

You can draw an ellipse by the following procedure.

1) Insert two tacks or drawing pins, A and B, as shown in the figure into a sheet of drawing paper at a distance 'd' apart.
2) Tie the two ends of a piece of thread whose length is greater than '2d' and place the loop around AB as shown in the figure.
3) Place a pencil inside the loop of thread, pull the thread taut and move the pencil sidewise, keeping the thread taut.

The pencil will trace an ellipse.



1 Law of orbit

All planets move in elliptical orbits around the Sun with the Sun at one of the foci of the ellipse.

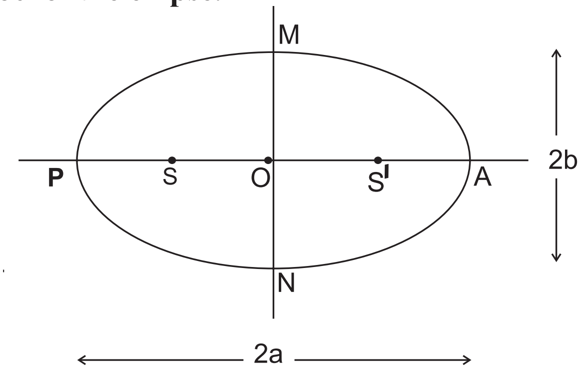


Fig. 5.1: An ellipse traced by a planet with the Sun at the focus.

The orbit of a planet around the Sun is shown in Fig. 5.1.

Here, S and S′ are the foci of the ellipse the Sun being at S.

P is the closest point along the orbit from S and is, called 'Perihelion'.

A is the farthest point from S and is, called 'Aphelion'.

PA is the major axis = 2a.

PO and AO are the semimajor axes = a.

MN is the minor axis =2b.

MO and ON are the semiminor axes = b

2. Law of areas

The line that joins a planet and the Sun sweeps equal areas in equal intervals of time.

Kepler observed that planets do not move around the Sun with uniform speed. They move faster when they are nearer to the Sun while they move slower when they are farther from the Sun. This is explained by this law.

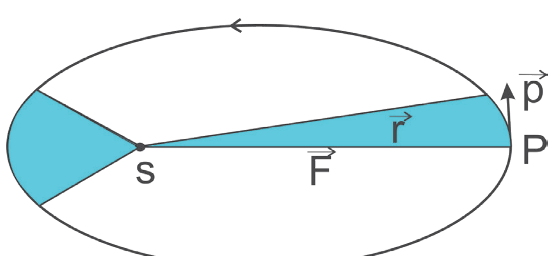


Fig. 5.2: The orbit of a planet P moving around the Sun.

Fig 5.2 shows the orbit of a planet. The shaded areas are the areas swept by SP, the line joining the planet and the Sun, in fixed intervals of time. These are equal according to the second law.

The law of areas can be understood as an outcome of conservation of angular momentum. It is valid for any central force. A central force on an object is a force which is always directed along the line joining the position of object and a fixed point usually taken to be the origin of the coordinate system. The force of gravity due to the Sun on a planet is always along the line joining the Sun and the planet (Fig. 5.2). It is thus a central force. Suppose the Sun is at the origin. The position of planet is denoted by $\vec{r}$ and the perpendicular component of its momentum is denoted by $\vec{p}$ (component $\perp \vec{r}$). The area swept by the planet of mass m in given interval Δt is $\overrightarrow{\Delta A}$ which is given by

$$\overrightarrow{\Delta A} = \frac{1}{2}(\vec{r} \times \vec{v}\Delta t) \qquad \text{--- (5.1)}$$

As for small Δt, $\vec{v}$ is perpendicular to $\vec{r}$ and this is the area of the triangle.

$$\therefore \frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{1}{2}(\vec{r} \times \vec{v}) \qquad \text{--- (5.2)}$$

Linear momentum ($\vec{p}$) is the product of mass and velocity.

$$\vec{p} = m\vec{v} \qquad \text{--- (5.3)}$$

∴ putting $\vec{v} = \vec{p}/m$ in the above equation, we get

$$\frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{1}{2}\left(\vec{r} \times \frac{\vec{p}}{m}\right) \qquad \text{--- (5.4)}$$

Angular momentum $\vec{L}$ is the rotational equivalent of linear momentum and is defined as

$$\therefore \vec{L} = \vec{r} \times \vec{p} \qquad \text{--- (5.5)}$$

For central force the angular momentum is conserved.

$$\therefore \frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{\vec{L}}{2m} = \text{constant} \qquad \text{--- (5.6)}$$

∴ This proves the law of areas. This is a consequence of the gravitational force being a central force.

3. Law of periods

The square of the time period of revolution of a planet around the Sun is proportional to the cube of the semimajor axis of the ellipse traced by the planet.

If r is length of semimajor axis then, this law states that

$$T^2 \propto r^3$$

$$\text{or} \quad \frac{T^2}{r^3} = \text{constant} \qquad \text{--- (5.7)}$$

Kepler's laws were based on regular observations of the motion of planets. Kepler did not know why the planets obey these laws,. i.e. he had not derived these laws.

Table 5.1 gives data from measurements of planetary motions which confirm Kepler's law of periods.

Table 5.1: Kepler's third law

Planet	Semi-major axis in units of 10^{10} m	Period in years	T^2/r^3 in units of 10^{-34} y^2m^{-3}
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

Example 5.1: What would be the average duration of year if the distance between the Sun and the Earth becomes

(A) thrice the present distance.

(B) twice the present distance.

Solution:

(A) Let r_1 = Present distance between the Earth and Sun

T = 365 days.

If $r_2 = 3r_1$, T_2 = ?

According to Kepler's law of period

$T_1^2 \propto r_1^3$ and $T_2^2 \propto r_2^3$

$$\therefore \frac{T_2^2}{T_1^2} = \frac{r_2^3}{r_1^3}$$

$$\therefore \frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2}$$

$$= \left(\frac{3r_1}{r_1}\right)^{3/2}$$

$$\therefore \frac{T_2}{T_1} = \sqrt{27}$$

$$T_2 = T_1 \times \sqrt{27}$$

$$= 365 \times \sqrt{27}$$

$$= 1897 \text{days}$$

(B) If $r_2 = 2r_1$, T_2 = ?

$$\frac{T_2^2}{T_1^2} = \frac{r_2^3}{r_1^3}$$

$$\frac{T_2^2}{T_1^2} = \left(\frac{2r_1}{r_1}\right)^3$$

$$\therefore \frac{T_2}{T_1} = \sqrt{8}$$

$$T_2 = T_1\sqrt{8}$$

$$= 365\sqrt{8}$$

=1032 days.

5.3 Universal Law of Gravitation:

When objects are released near the surface of the Earth, they always fall down to the ground, i.e., the Earth attracts objects towards itself. Galileo (1564-1642) pointed out that heavy and light objects, when released from the same height, fall towards the Earth at the same speed, i.e., they have the same acceleration. Newton went beyond (the Earth and objects falling on it) and proposed that the force of attraction between masses is universal. Newton stated the universal law of gravitation which led to an explanation of terrestrial gravitation. It also explains the Kepler's laws and provides the reason behind the observed motion of planets around the Sun.

In 1665, Newton studied the motion of moon around the Earth. It was known that the moon completes one revolution about the Earth in 27.3 days. The distance from the Earth to the moon is 3.85×10^5 km. The motion of the moon is in almost a circular orbit around the Earth with constant angular speed ω. As it is a circular motion, the moon must be constantly acted upon by a force directed towards the Earth which is at the centre of the circle. This force is the centripetal force, and is given by

$$F = mr\omega^2 \qquad \text{--- (5.8)}$$

where m is the mass of the moon and r is the distance between the centres of the moon and the Earth.

Also we have $F = ma$ from Newton's laws of motion.

$$\therefore ma = mr\omega^2$$

$$\therefore a = r\omega^2 \qquad \text{--- (5.9)}$$

As angular velocity in terms of time period is given as

$$\omega = \frac{2\pi}{T}$$

we get

$$a = r\left(\frac{2\pi}{T}\right)^2 \qquad \text{--- (5.10)}$$

Substituting values of r and T, we get

$$a = \frac{3.85\times10^5\times10^3 4\pi^2 \text{m}}{(27.3\times24\times60\times60)^2\ \text{s}^2}$$

$$\therefore a = 0.0027\ \text{m}/\text{s}^2$$

This is the acceleration of the moon which is towards the centre of the Earth, i.e., centre of orbit in which the moon revolves. What could be the force which produces this acceleration?

Do you know ?

The value of acceleration due to gravity can be assumed to be constant when we are dealing with objects close to the surface of the Earth. This is because the difference in their distances from the centre of the Earth is negligible.

Newton assumed that the laws of nature are the same for Earthly objects and for celestial bodies. As this acceleration is much smaller than the acceleration felt by bodies near the surface of the Earth (while falling on Earth), he concluded that the acceleration felt by an object due to the gravitational force of the Earth must be decreasing with distance of the body from the Earth. (Remember that the value of acceleration due to Earth's gravity at the surface is 9.8 m/s^2)

We have,

$$\frac{a_{object}}{a_{moon}} = \frac{9.8\ \text{m}/\text{s}^2}{0.0027\ \text{m}/\text{s}^2} \approx 3600$$

Also,

$$\frac{\text{distance of moon from the Earth's centre}}{\text{distance of object from the Earth's centre}}$$

$$= \frac{3.85\times10^5\ \text{km}}{6378\ \text{km}} \approx 60$$

Thus from the above two equations we get

$$\therefore \frac{a_{object}}{a_{moon}} = \left[\frac{\text{distance of moon}}{\text{distance of object}}\right]^2 \qquad \text{--- (5.11)}$$

Newton therefore concluded that the acceleration of an object towards the Earth is inversely proportional to the square of distance of object from the centre of the Earth.

$$\therefore a \propto \frac{1}{r^2}$$

As, $\quad F = ma$

Therefore, the force exerted by the Earth on an object of mass m at a distance r from it is

$$F \propto \frac{m}{r^2}$$

Similarly an object also exerts a force on the Earth which is

$$F_E \propto \frac{M}{r^2}$$

where M is the mass of the Earth.

According to Newton's third law of motion, the force on a body due to the Earth has to be equal to the force on the Earth due to the object. Hence the force F is also proportional to the mass of the Earth. Hence Newton concluded that the gravitational force between the Earth and an object of mass m is

$$\text{F} \propto \frac{\text{Mm}}{\text{r}^2}$$

He then generalized it to gravitational force between any two objects and stated his

Universal law of gravitation as follows.

Every particle of matter attracts every other particle of matter with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

This law is applicable to all material objects in the universe. Hence it is known as the universal law of gravitation.

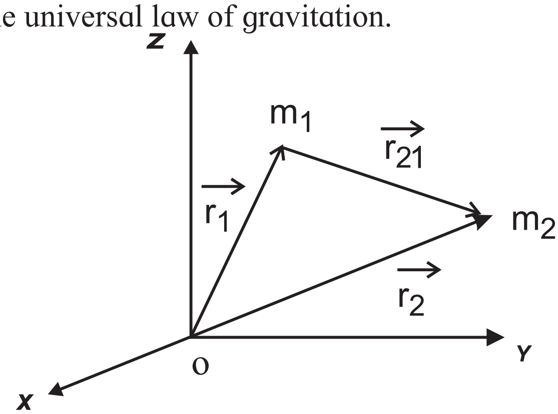


Fig. 5.3: Gravitational force between masses m_1 and m_2.

If two bodies of masses m_1 and m_2 are separated by a distance r, then the gravitational force of attraction between them can be written as

$$F \propto \frac{m_1 m_2}{r^2}$$

$$\text{or,} \quad F = G\frac{m_1 m_2}{r^2} \qquad \text{--- (5.12)}$$

where G is a constant known as the universal gravitational constant. Its value in SI units is given by

$$G = 6.67\times10^{-11}\text{N m}^2/\text{kg}^2$$

and its dimensions are

$$[G] = [L^3M^{-1}T^{-2}].$$

The gravitational force is an attractive force and it acts along the line joining the two bodies. The forces exerted by two bodies on each other have same magnitude but have opposite directions, they form an action-reaction pair.

Example 5.2: The gravitational force between two bodies is 1 N. If distance between them is doubled, what will be the gravitational force between them?

Solution: Let the masses of the two bodies be m_1 and m_2 and the distance between them be r.

The force between them, $F_1 = \frac{Gm_1m_2}{r^2}$

When the distance between them is doubled the force becomes, $F_2 = \frac{Gm_1m_2}{(2r)^2} = \frac{Gm_1m_2}{4r^2}$

$$\therefore \frac{F_1}{F_2} = \frac{Gm_1m_2}{r^2} \times \frac{4r^2}{Gm_1m_2}$$

$$\therefore \frac{F_1}{F_2} = 4$$

$$\therefore F_2 = \frac{1}{4}\text{N} \quad (\because F_1 = 1\text{N})$$

$$F_2 = 0.25\text{ N}$$

$\therefore$ Force become one forth (0.25 N)

Figure 5.3 shows two point masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively from origin O. The position vector of m_2 with respect to m_1 is then given by $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$. Similarly $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = -\vec{r}_{21}$ and if $r = |\vec{r}_{12}| = |\vec{r}_{21}|$, the formula for force on m_2 due to m_1 can be expressed in vector form as,

$$\vec{F}_{21} = G\frac{m_1m_2}{r^2}(-\hat{r}_{21}) \qquad \text{--- (5.13)}$$

where $\hat{r}_{21}$ is the unit vector from m_1 to m_2. The force $\vec{F}_{21}$ is directed from m_2 to m_1. Similarly, force experienced by m_1 due to m_2 is $\vec{F}_{12}$

$$\vec{F}_{12} = G\frac{m_1m_2}{r^2}(-\hat{r}_{12}) \qquad \text{--- (5.14)}$$

$$\therefore \vec{F}_{12} = -\vec{F}_{21} \qquad \text{--- (5.15)}$$

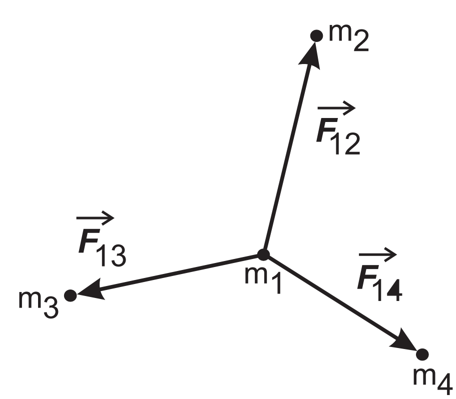


Fig. 5.4: gravitational force due to a collection of masses.

This law refers to two point masses. For a collection of point masses, the force on any one of them is the vector sum of the gravitational forces exerted by all the other point masses. As shown in Fig. 5.4, the resultant force on point mass m_1 is the vector sum of forces $\vec{F}_{12}$, $\vec{F}_{13}$ and $\vec{F}_{14}$ due to point masses m_2, m_3 and m_4 respectively. Masses m_2, m_3 and m_4 are also attracted towards mass m_1 and there is also mutual attraction between masses m_2, m_3 and m_4 but these forces are not shown in the figure.

For n particles, force on ith mass $\vec{F}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{F}_{ij}$

where $\vec{F}_{ij}$ is the force on ith particle due to jth particle.

The gravitational force between an extended object like the Earth and a point mass A can be obtained by obtaining the vector sum of forces on the point mass A due to each of the point mass which make up the extended object. We can consider the following two special cases, for which we can get a simple result. We will state the result here and show how it can be understood qualitatively.

(1) The gravitational force of attraction due to a hollow, thin spherical shell of uniform density, on a point mass situated inside it is zero.

This can be qualitatively understood as follows. First let us consider the case when the point mass A, is at the centre of the hollow thin shell. In this case as every point on the shell is equidistant from A, all points exert force of equal magnitude on A but the directions of these forces are different. Now consider the forces on A due to two diametrically opposite points on the shell. The forces on A due to them will be of equal magnitude but will be in opposite directions and will cancel each other. Thus forces due to all pairs of points diametrically opposite to each other will cancel and there will be no net force on A due to the shell. When the point object is situated elsewhere inside the shell, the situation is not so symmetric. Gravitational force varies directly with mass and inversely with square of the distance. Some part of the shell may be closer to point A, but its mass is less. Remaining part will then have larger mass but its centre of mass is away from A. However mathematically it can be shown that the net gravitational force on A is still zero, so long as it is inside the shell. In fact, the gravitational force at any point inside any hollow closed object of any shape is zero.

(2) The gravitational force of attraction between a hollow spherical shell or solid sphere of uniform density and a point mass situated outside is just as if the entire mass of the shell or sphere is concentrated at the centre of the shell or sphere.

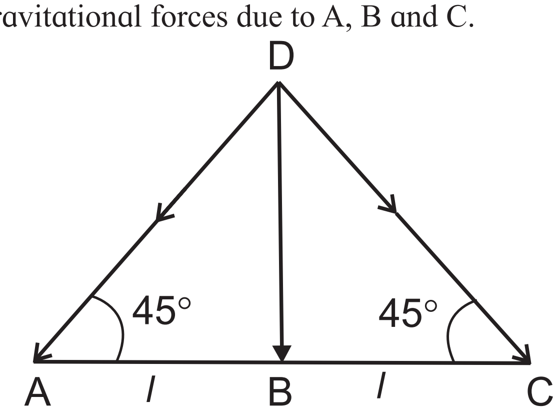
Gravitational force caused by different regions of shell can be resolved into components along the line joining the point mass to the centre and along a direction perpendicular to this line. The components perpendicular to this line cancel each other and the resultant force remains along the line joining the point to the centre. By mathematical calculations it can be shown to be equal to the force that would have been exerted if the entire mass of the shell was present at the centre of the shell.

It is obvious that case (2) is applicable for any uniform sphere (solid or hollow), so long as the point is outside the sphere.

Example 5.3: Three particles A, B, and C each having mass m are kept along a straight line with AB = BC = l. A fourth particle D is kept on the perpendicular bisecter of AC at a distance l from B. Determine the gravitational force on D.

Solution : CD = AD = $\sqrt{AB^2 + BD^2} = \sqrt{2}\, l$

Gravitational force on D = Vector sum of gravitational forces due to A, B and C.



Force due to A = $\frac{Gmm}{(AD)^2}=\frac{Gm^2}{2l^2}$. This will be along $\overrightarrow{DA}$

Force due to C = $\frac{Gmm}{(CD)^2}=\frac{Gm^2}{2l^2}$. This is along $\overrightarrow{DC}$

Force due to B = $\frac{Gmm}{(BD)^2}=\frac{Gm^2}{l^2}$. This is along $\overrightarrow{DB}$

We can resolve the forces along horizontal and vertical directions.

Let the unit vector along horizontal direction $\overrightarrow{AC}$ be $\hat{i}$ and along the vertical direction $\overrightarrow{BD}$ be $\hat{j}$

Net horizontal force on D

$$=\frac{Gm^2}{2l^2}\cos 45°(-\hat{i})+\frac{Gm^2}{l^2}\cos 90°(\hat{i})+\frac{Gm^2}{2l^2}\cos 45°(\hat{i})$$

$$=\frac{-Gm^2}{2\sqrt{2}l^2}+\frac{Gm^2}{2\sqrt{2}l^2}=0$$

Net vertical force on D

$$=\frac{Gm^2}{2l^2}\cos 45°(-\hat{j})+\frac{Gm^2}{l^2}(-\hat{j})+\frac{Gm^2}{2l^2}\cos 45°(-\hat{j})$$

$$=\frac{-Gm^2}{l^2}\left(\frac{1}{\sqrt{2}}+1\right)(\hat{j})$$

$$=\frac{Gm^2}{l^2}\left(\frac{1}{\sqrt{2}}+1\right)(-\hat{j})$$

$(-\hat{j})$ shows that the net force is directed along DB

5.4 Measurement of the Gravitational Constant (*G*):

The magnitude of the gravitational constant *G* can be found by measuring the force of gravitational attraction between two bodies of masses m_1 and m_2 separated by certain distance '*L*'.This can be measured by using the Cavendish balance.

The Cavendish balance consists of a light rigid rod. It is supported at the centre by a fine vertical metallic fibre about 100 cm long. Two small spheres s_1 and s_2 of lead having equal mass *m* and diameter about 5 cm are mounted at the ends of the rod and a small mirror M is fastened to the metallic fibre as shown in Fig. 5.5. The mirror can be used to reflect a beam of light onto a scale and thereby measure the angel through which the wire will be twisted.

Two large lead spheres L_1 and L_2 of equal mass *M* and diameter of about 20 cm are brought close to the small spheres on opposite side as shown in Fig. 5.5. The big spheres attract the nearby small spheres by equal and opposite force. Let $\vec{F}$ be the force of attraction between a big sphere and small sphere near to it. Hence a torque will be generated without exerting any net force on the bar. Due to this torque the bar turns and the suspension wire gets twisted till the restoring torque due to the elastic property of the wire becomes equal to the gravitational torque.

The gravitational force between the spherical balls is the same as if their masses are concentrated at their centres. If *r* is the initial distance of separation between the centres of the big and the neighbouring small sphere, then the magnitude of the force between them is

$$F=G\frac{mM}{r^2}$$

If length of the rod is L, then the magnitude of the torque arising out of these forces is

$$\tau=FL=G\frac{mM}{r^2}L \quad \text{--- (5.16)}$$

At equilibrium, it is equal and opposite to the restoring torque.

$$\therefore G\frac{mM}{r^2}L=K\theta \quad \text{--- (5.17)}$$

where *K* is the restoring torque per unit angle and θ is the angle of twist.

By applying a known torque τ_1 and measuring the corresponding angle of twist α, the restoring torque per unit twist can be determined as $K=\tau_1/\alpha$.

Thus, in actual experiment measuring θ and knowing values of τ, *m, M* and *r*, the value of *G* can be calculated from Eq. (5.17). The gravitational constant measured in this way is found to be

$$G = 6.67\times10^{-11} \text{N m}^2/\text{kg}^2$$

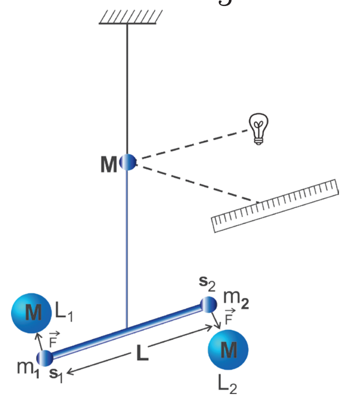


Fig 5.5 : The Cavendish balance.

5.5 Acceleration due to Gravity:

We have seen in section 5.3 that the magnitude of the gravitational force on a point object of mass m due to another point object of mass M at a distance r from it is given by the equation.

$$F = G\frac{mM}{r^2}$$

This formula can be used to calculate the gravitational force on an object due to the Earth. We know that the Earth is an extended object. In many practical applications Earth can be assumed to be a uniform sphere. As seen in section 5.3 its entire mass can be assumed to be concentrated at is centre. Thus if the mass of the Earth is M and that of the point object is m and the distance of the point object from the centre of the Earth is r then the force of attraction between them is given by

$$F = G\frac{Mm}{r^2}$$

If the point object is not acted upon by any other force, it will be accelerated towards the centre of the Earth under the action of this force. Its acceleration can be calculated by using Newton's second law $F = ma$.

Acceleration due to the gravity of the Earth =

$$G\frac{Mm}{r^2}\times\frac{1}{m}$$

$$= \frac{GM}{r^2} \qquad \text{--- (5.18)}$$

This is known as the acceleration due to gravity of the Earth and denoted by g.

If the object is close to the surface of the Earth, $r \cong R$, the radius of the Earth then

$$g_{\text{Earth's surface}} = \frac{GM}{R^2} \qquad \text{--- (5.19)}$$

Example 5.4: Calculate mass of the Earth from given data,

Acceleration due to gravity $g = 9.81 \text{m/s}^2$

Radius of the Earth $R_E = 6.37\times10^6 \text{ m}$

$G = 6.67\times10^{-11} \text{ N m}^2/\text{kg}^2$

Solution:

$$g = \frac{GM_E}{R_E^2}$$

$$\therefore M_E = \frac{gR_E^2}{G}$$

$$\therefore M_E = \frac{9.81\times(6.37\times10^6)^2}{6.67\times10^{-11}}$$

$$\therefore M_E = 5.97\times10^{24} \text{ kg}$$

The value of g depends only on the properties of the Earth and does not depend on the mass of the object. This is exactly what Galileo had found from his experiments of dropping objects with different masses from the same height.

Do you know ?

An object of mass m (much smaller than the mass of the Earth) is attracted towards the Earth and falls on it. The Earth is also attracted by the same force (magnitude) toward the mass m. However, its acceleration towards m will be

$$a_{earth} = \frac{\left(G\frac{Mm}{r^2}\right)}{M} = \frac{Gm}{r^2}$$

$$\therefore \frac{a_{earth}}{g} = \frac{m}{M} \quad \left(\text{as } g = \frac{GM}{r^2}\right)$$

As $m << M$, $a_{Earth} << g$ and is nearly zero. Thus, practically only the mass m moves towards the Earth.

Example 5.5: Calculate the acceleration due to gravity on the surface of moon if mass of the moon is 1/80 times that of the Earth and diameter of the moon is 1/4 times that of the Earth (g =9.8 m/s^2)

Solution:

M_m = Mass of the moon = $M/80$,

where M is mass of the Earth.

R_m = Radius of the moon = $R/4$,

where R is Radius of the Earth.

Acceleration due to gravity on the surface of the Earth, $g = GM/R^2$ --- (1)

Acceleration due to gravity on the surface of the moon, $g_m = GM_m/R_m^2$ --- (2)

$\therefore$ From equation (1) and (2)

$$\frac{g_m}{g} = \frac{M_m}{M} \times \left[\frac{R}{R_m}\right]^2$$

$$\frac{g_m}{g} = \frac{1}{80} \times \left[\frac{4}{1}\right]^2$$

$$\therefore \frac{g_m}{g} = \frac{1}{5}$$

$$\therefore g_m = \frac{g}{5} = \frac{9.8}{5}$$

$$\therefore g_m = 1.96 \text{ m/s}^2$$

Example 5.6: Find the acceleration due to gravity on a planet that is 10 times as massive as the Earth and with radius 20 times of the radius of the Earth ($g = 9.8$ m/s^2).

Solution : Let mass of the planet be M_p, radius of the Earth and that of the planet be R_E and R_P respectively. Let mass of the Earth be M_E and g_p be acceleration due to gravity on the planet.

$$M_p = 10\, M_E$$

$$R_p = 20\, R_E,\ g = 9.8 \text{ m/s}^2$$

$$g_p = ?$$

$$g = \frac{GM_E}{R_E^2},\ g_P = \frac{GM_P}{R_P^2}$$

$$\therefore g_P = \frac{G(10M_E)}{(20R_E)^2}$$

$$= \frac{10GM_E}{400R_E^2}$$

$$= \frac{1}{40}g$$

$$= 0.245 \text{ m/s}^2$$

5.6 Variation in the Acceleration due to Gravity with Altitude, Depth, Latitude and Shape:

(A) Variation in g with Altitude:

Consider a body of mass m on the surface of the Earth. The acceleration due to gravity on the Earth's surface is given by,

$$g = \frac{GM}{R^2}$$

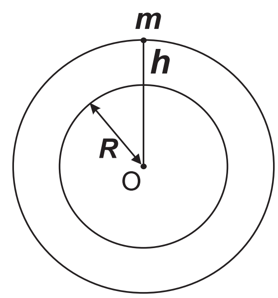


Fig. 5.6 Acceleration due to gravity at height h above the Earth's surface.

When the body is at height h above the surface of the Earth as shown in Fig. 5.6, acceleration due to gravity changes to

$$g_h = \frac{GM}{(R+h)^2}$$

$$\therefore \frac{g_h}{g} = \frac{\frac{GM}{(R+h)^2}}{\frac{GM}{R^2}}$$

$$\therefore \frac{g_h}{g} = \frac{R^2}{(R+h)^2}$$

$$\therefore g_h = \frac{g\,R^2}{(R+h)^2} \quad \text{--- (5.20)}$$

This equation shows that, the acceleration due to gravity goes on decreasing with increase in altitude of body from the surface of the Earth. We can rewrite Eq. (5.20) as

$$g_h = \frac{g\,R^2}{R^2\left(1+\frac{h}{R}\right)^2}$$

$$\therefore g_h = g\left(1+\frac{h}{R}\right)^{-2}$$

For small altitude h, i.e., for $\frac{h}{R} << 1$,

$$\therefore g_h \simeq g\left(1-\frac{2h}{R}\right) \quad \text{--- (5.21)}$$

(by neglecting higher power terms of $\frac{h}{R}$ as $\frac{h}{R} << 1$)

This expression can be used to calculate the value of g at height h above the surface of the Earth as long as $h << R$.

Example 5.7 : At what distance above the surface of Earth the acceleration due to gravity decreases by 10% of its value at the surface? (Radius of Earth = 6400 km). Assume the distance above the surface to be small compared to the radius of the Earth.

Solution : g_h = 90% of g (g decreases by 10% hence it becomes 90%)

or, $$\frac{g_h}{g} = \frac{90}{100} = 0.9$$

From Eq. (5.21)

$$g_h = g\left[1 - \frac{2h}{R}\right]$$

$$\therefore \frac{g_h}{g} = 1 - \frac{2h}{R}$$

$$\therefore 0.9 = 1 - \frac{2h}{R}$$

$$h = \frac{R}{20}$$

$$h = 320 \text{ km}$$

(B) Variation in g with Depth:

The Earth can be imagined to be a sphere made of large number of concentric uniform spherical shells. The total mass of the Earth is the combined mass of all the shells. When an object is on the surface of the Earth it experiences the gravitational force as if the entire mass of the Earth is concentrated at its centre.

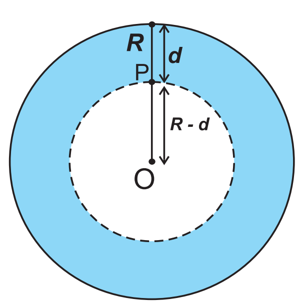


Fig. 5.7 Acceleration due to gravity at depth d below the surface of the Earth.

The acceleration due to gravity according to eq. (5.19) is

$$g = \frac{GM}{R^2}$$

Assuming that the density of the Earth is uniform, it is given by

$$\rho = \frac{\text{Mass } (M)}{\text{Volume} (V)}$$

$$\therefore M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore g = \frac{G \times \frac{4}{3}\pi R^3 \rho}{R^2}$$

$$\therefore g = \frac{4}{3}\pi R \rho G \quad \text{--- (5.22)}$$

Consider a body at a point P at the depth d below the surface of the Earth as shown in Fig. 5.7. Here the force on a body at P due to the material outside the inner sphere shown by shaded region, can be shown to cancel out due to symmetry. The net force on P is only due to the material inside the inner sphere of radius OP = $R - d$. Acceleration due to gravity because of this sphere is

$$g_d = \frac{GM'}{(R-d)^2}$$

where M' = volume of the inner sphere×density

$$\therefore M' = \frac{4}{3}\pi (R-d)^3 \times \rho$$

$$\therefore g_d = \frac{G \times \frac{4}{3}\pi (R-d)^3 \rho}{(R-d)^2}$$

$$\therefore g_d = G \times \frac{4}{3}\pi (R-d)\rho \quad \text{--- (5.23)}$$

Dividing Eq. (5.23) by Eq. (5.22) we get,

$$\therefore \frac{g_d}{g} = \frac{R-d}{R}$$

$$\therefore \frac{g_d}{g} = 1 - \frac{d}{R}$$

$$\therefore g_d = g\left[1 - \frac{d}{R}\right] \quad \text{--- (5.24)}$$

This equation gives acceleration due to gravity at depth d below the Earth's surface.

It shows that the acceleration due to gravity decreases with depth.

Special case :

At the centre of the Earth, where $d = R$, Eq. (5.24) gives $g_d = 0$

Hence, a body of mass m if taken to the centre of Earth, will not experience the force of gravity due to the Earth. This can also be understood to be due to symmetry. The case is similar to the force of gravity on an object placed at the centre of a spherical shell as seen in section 5.3.

Thus, the value of acceleration due to gravity is maximum at the surface of the Earth. The value goes on decreasing with

1) increase in depth below the Earth's surface. (varies linearly with $(R\text{-}d) = r$)

2) increase in height above the Earth's surface. (varies inversely with $(R+h)^2 = r^2$)

Graphically the variation of acceleration due to gravity according to depth and height can be expressed as follows. We have plotted the value of g as a function of r, the distance from the centre of the Earth, in Fig. 5.8. For $r < R$ i. e. below the surface of the Earth, we use Eq. (5.24), according to which $g_d = g\left(1 - \dfrac{d}{R}\right)$

Writing $R - d = r$, the distance from the centre of the Earth, we get the value of g as a function of r, $g(r) = g\dfrac{r}{R}$ which is the equation of a straight line with slope g/R and passing through the origin.

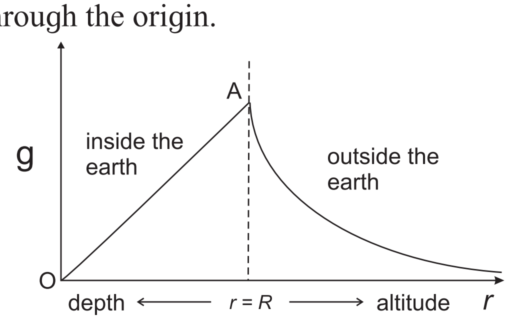


Fig 5.8 - Variation of g due to depth and altitude from the Earth's surface.

For $r > R$ we have to use Eq. (5.20). Writing $R + h = r$ we have

$g(r) = \dfrac{gR^2}{r^2}$ which is plotted in Fig. 5.8

(C) Variation in g with Latitude and Rotation of the Earth:

Latitude is an angle made by radius vector of any point from centre of the Earth with the equatorial plane. Obviously it ranges from 0^0 at the equator to 90^0 at the poles.

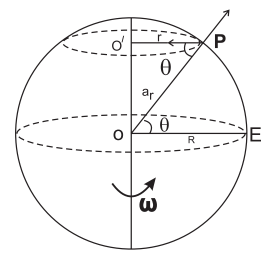


Fig. 5.9 Variation of g with latitude.

The Earth rotates about its polar axis from west to east with uniform angular velocity ω. Hence every point on the surface of the Earth (except the poles) moves in a circle parallel to the equator. The motion of a mass m at point P on the Earth is shown by the dotted circle with centre at O′ in Fig. 5.9. Let the latitude of P be θ and radius of the circle be r.

$PO' = r$

$\angle EOP = \theta$, E being a point on the equator

$\therefore \angle OPO' = \theta$

In Δ OPO′, $\cos\theta = \dfrac{PO'}{PO} = \dfrac{r}{R}$

$\therefore r = R\cos\theta$

The centripetal acceleration for the mass m, directed along PO′ is

$a = r\omega^2$

$a = R\omega^2\cos\theta$

The component of this centripetal acceleration along PO, i.e., towards the centre of the Earth is

$a_r = a\cos\theta$

$\therefore a_r = R\omega^2\cos\theta.\cos\theta$

$a_r = R\omega^2\cos^2\theta$

Part of the gravitational force of attraction on P acting towards PO is utilized in providing this components of centripetal acceleration.

Thus the effective force of gravitational attraction on m at P can be written as

$mg' = mg - mR\omega^2\cos^2\theta$

g' being the effective acceleration due to gravity at P i.e., at latitude θ. This is thus given by $g' = g - R\omega^2\cos^2\theta$ --- (5.25)

As the value of θ increases, cosθ decreases. Therefore g' will increase as we move away from equator towards any pole due to the rotation of the Earth.

special case I At equator $\theta = 0$

$$\cos\theta = 1$$

$$g' = g - R\omega^2$$

The effective acceleration due to gravity is minimum at equator, as here it is reduced by maximum amount. The reduction here is $g - g' = R\omega^2$

$R = 6.4 \times 10^6$ m ---Radius of the Earth and

ω = Angular velocity of rotation of the Earth

$$\omega = \frac{2\pi}{T}$$

$$\therefore \omega = \frac{2\pi}{24 \times 60 \times 60}$$

$$\therefore \omega = 7.275 \times 10^{-5}\,\mathrm{s}^{-1}$$

$$\therefore g - g' = R\omega^2$$

$$\therefore g - g' = 0.03386\ \mathrm{m/s^2}$$

Case II At poles $\theta = 90^0$

$$\cos\theta = 0$$

$$\therefore g' = g - R\omega^2\cos\theta$$

$$= g - 0$$

$$= g$$

There is no reduction in acceleration due to gravity at poles, due to the rotation of the Earth as the poles are lying on the axis of rotation and do not revolve.

Variation of g with latitudes at sea level is given in the following table.

Table 5.2: Variation of g with latitude

Latitude (°)	g (m/s²)
0	9.7804
10	9.7819
20	9.7864
30	9.7933
40	9.8017
50	9.8107
60	9.8192
70	9.8261
80	9.8306
90	9.8322

Effect of the shape of the Earth: Quite often we assume the Earth to be a sphere. However, it is actually on ellipsoid; bulged at equator. Hence equatorial radius of Earth (6378 km) is greater than the polar radius (6356 km). Thus, on the equator, there is combined effect of greater radius and rotation in reducing the force of gravity. As a result, the acceleration due to gravity on the equator is $g_E = 9.7804$ m/s² and on the poles it is $g = 9.8322$ m/s².

Weight of an object is the force with which the Earth attracts that object. Thus, weight $w = mg$ where m is the mass of the object. As the value of g changes with altitude, depth and latitude, the weight also changes. Weight of an object is minimum at the equator. Similarly, the weight of an object reduces with increasing height above the Earth's surface and with increasing depth below the its surface.

5.7 Gravitational Potential and Potential Energy:

In earlier standards, you have studied potential energy as the energy possessed by an object on account of its position or configuration. The word *configuration* corresponds to the distribution of the particles in the object. More specifically, potential energy is the work done *against conservative force* (or forces) *in achieving* a certain position or configuration *of a given system*. It always depends upon the relative positions of the particles in that system. There is a universal principle that states, **Every system always configures itself in order to have minimum potential energy or every system tries to minimize its potential energy.** Obviously, in order to change the configuration, you will have to do work.

Examples:

(I) A spring in its natural state, possesses minimum potential energy. Whenever we stretch it or compress it, we perform work against the conservative force (in this case, the elastic restoring force). Due to this work, the relative distances between the particles of the system change (configuration changes) and its potential energy increases. The spring finally regains its original configuration of minimum potential energy on removal of the applied force.

(II) When an object is lying on the Earth, the system of that object and the Earth has minimum potential energy. This is the gravitational potential energy of the system as these two are bound by the gravitational force. While lifting the object to some height (new position), we do work against the conservative gravitational force in order to achieve the new position.

In its new position, the object is at rest due to balanced forces. If you are holding the object, the force of static friction between the object and your fingers balances the gravitational force. If kept on a surface, the normal reaction force given by the surface balances the gravitational force. However, now, the object has a ***capacity*** to acquire kinetic energy, when given an opportunity (when allowed to fall). We call this increase in the capacity as the potential energy *gained* by the system. As we raise it more and more, this capacity, and hence potential energy of the system, increases. It falls on the Earth to achieve the configuration of minimum potential energy on dropping it from the new position.

Thus, in general, we can write **work done against a conservative force acting on an object = Increase in the potential energy of the system.**

$$\therefore \vec{F}.d\vec{x} = dU$$

Here dU is the change in potential energy while displacing the object through $d\vec{x}$, $\vec{F}$ being the force acting on the object

It should be remembered that potential energy is *always* of the system as a whole. For an object on the Earth, it is of the system of the object and the Earth and not only of that object. There is no meaning to potential energy of an isolated object in the intergalactic (gravity free) space, in the absence of any conservative force acting upon it.

5.7.1 Expression for Gravitational Potential Energy:

Work done against gravitational force $\vec{F}_g$, in displacing an object through a small displacement $d\vec{r}$, appears as increase in the potential energy of the system.

$$\therefore dU = -\vec{F}_g.d\vec{r}$$

Negative sign appears because dU is the work done by us (external agent) *against* the gravitational force $\vec{F}_g$

For displacement of the object from an initial position $\vec{r}_i$ to the final position $\vec{r}_f$, the change in potential energy ΔU, can be obtained by integrating dU.

$$\therefore \Delta U = \int_{r_i}^{r_f} (dU) = \int_{r_i}^{r_f} \left(-\vec{F}_g.d\vec{r}\right)$$

Gravitational force of the Earth, $\vec{F}_g = -\frac{GMm}{r^2}\hat{r}$ where $\hat{r}$ is the unit vector in the direction of $\vec{r}$. Negative sign appears here because $\vec{r}$ is from centre of the Earth to the object and $\vec{F}_g$ is directed towards centre of the Earth.

∴ For 'Earth and mass' system,

$$\Delta U = \int_{r_i}^{r_f} dU = \int_{r_i}^{r_f} -\left(-\frac{GMm}{r^2}\hat{r}\right).d\vec{r}$$

$$= GMm\int_{r_i}^{r_f}\left(\frac{dr}{r^2}\right) \text{ as } d\vec{r} \text{ is along } \hat{r}$$

$$= GMm\left(-\frac{1}{r}\right)_{r_i}^{r_f}$$

$$= GMm\left(\frac{1}{r_i} - \frac{1}{r_f}\right) \quad \text{---(5.26)}$$

Change in potential energy corresponds to the work done against conservative forces. The absolute value of potential energy is not defined. It is logical as well as convenient to choose the point of zero potential energy to be the point of zero force. For gravitational force, such point is taken at $r = \infty$. This point should be chosen as the initial point so that initially the potential energy is zero. $\therefore U(r_i) = 0$ at $r_i = \infty$

Final point r_f is obviously the point where we need to determine the potential energy of the system. $\therefore r_f = r$

$$\therefore U(\mathrm{r})_{grav.} = GMm\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$= GMm\left(\frac{1}{\infty} - \frac{1}{r}\right)$$

$$= -\frac{GMm}{r} \quad \text{--- (5.27)}$$

This is gravitational potential energy of the system of object of mass m and Earth of mass M having separation r (between their centres of mass).

Example 5.8: What will be the change in potential energy of a body of mass m when it is raised from height R_E above the Earth's surface to 5/2 R_E above the Earth's surface? R_E and M_E are the radius and mass of the Earth respectively.

Solution:

$$\Delta U = GmM_E\left(\frac{1}{r_i}-\frac{1}{r_f}\right)$$

$$= GmM_E\left(\frac{1}{2R_E}-\frac{1}{3.5R_E}\right)$$

$$= \frac{GmM_E}{R_E}\times\frac{1.5}{2\times 3.5} = \frac{2.14\,GmM_E}{R_E}$$

5.7.2 Connection of potential energy formula with *mgh*:

If the object is on the surface of Earth, $r = R$

$$\therefore U_1 = -\frac{GMm}{R}$$

If the object is lifted to height h above the surface of Earth, the potential energy becomes

$$U_2 = -\frac{GMm}{R+h}$$

Increase in the potential energy is given by

$$\Delta U = U_2 - U_1$$

$$= GMm\left(-\frac{1}{R+h}-\left[-\frac{1}{R}\right]\right)$$

$$= GMm\left(\frac{h}{R(R+h)}\right)$$

$$= \frac{GMmh}{R(R+h)}$$

If g is acceleration due to the Earth on the surface of Earth, $GM = gR^2$

$$\therefore \Delta U = mgh\left(\frac{R}{R+h}\right) \quad \text{--- (5.28)}$$

Eq. (5.28) gives the work to be done (or energy to be supplied) to raise an object of mass m to a height h, above the surface of the Earth.

If $h \ll R$, we can use $R+h \cong R$. Only in this case $\Delta U = mgh$ --- (5.29)

Thus, mgh is increase in the gravitational potential energy of the Earth -mass system if an object of mass m is lifted to a height h, provided h is negligible compared to radius of the Earth (up to a few kilometers).

5.7.3 Concept of Potential:

From eq. (5.27), the gravitational potential energy of the system of Earth and ***any*** mass m at a distance r from the centre of the Earth is given by

$$U = -\frac{GMm}{r}$$

$$= \left(-\frac{GM}{r}\right)m$$

$$= \left[\left(V_E\right)_r\right]m \quad \text{--- (5.30)}$$

The factor $-\frac{GM}{r} = \left(V_E\right)_r$ depends only upon mass of Earth and the location. Thus, it is the same for any mass m bound to the Earth. Conveniently, this is defined as the gravitational potential of Earth at distance r from its centre. In terms of potential, we can write the potential energy of the Earth-mass system as

Gravitational potential energy, U=Gravitational potential $V_r \times$ mass m **or** Gravitational potential is Gravitational potential energy per unit mass, i.e., $V_r = \frac{U}{m}$. The concept of potential can be defined on similar lines for any conservative force field.

Gravitational potential difference between any two points in gravitational field can be written as

$$V_2 - V_1 = \left(\frac{U_2-U_1}{m}\right) = \frac{dW}{m} \quad \text{--- (5.31)}$$

= Work done (or change in potential energy) per unit mass

In general, for a system of any two masses m_1 and m_2, separated by r, we can write

Gravitational potential energy,

$$U = -\frac{Gm_1m_2}{r} = \left(V_1\right)m_2 = \left(V_2\right)m_1 \quad \text{---(5.32)}$$

Here V_1 and V_2 are gravitational potentials at r due to m_1 and m_2 respectively.

5.7.4 Escape Velocity:

When any object is thrown vertically up, it falls back to the Earth after reaching a certain height. Higher the speed with which the object is thrown up, greater will be the height. If we keep on increasing the velocity, a stage will come when the object will reach heights so large that it will escape the gravitational field of the Earth and will not fall back on the Earth. This initial velocity is called the escape velocity.

Thus, the minimum velocity with which a body should be thrown vertically upwards from the surface of the Earth so that it escapes the Earth's gravitational field, is called the escape velocity (v_e) of the body. Obviously, as the gravitational force due to Earth becomes zero only at infinite distance, the object has to reach infinite distance in order to escape.

Let us consider the kinetic and potential energies of an object thrown vertically upwards with escape velocity v_e, when it is at the surface of the Earth and when it reaches infinite distance. On the surface of the Earth,

$$K.E. = \frac{1}{2}mv_e^2$$

$$P.E. = -\frac{GMm}{R}$$

Total energy = $P.E. + K.E.$

$$= \frac{1}{2}mv_e^2 - \frac{GMm}{R} \qquad \text{--- (5.33)}$$

The kinetic energy of the object will go on decreasing with time as it is pulled back by Earth's gravitational force. It will become zero when it reaches infinity. Thus at infinite distance from the Earth

$$K.E. = 0$$

Also, $P.E. = -\dfrac{GMm}{\infty} = 0$

$\therefore$ Total energy = $P.E. + K.E. = 0$

As energy is conserved

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0$$

$$\text{or, } v_e = \sqrt{\frac{2GM}{R}} \qquad \text{--- (5.34)}$$

Using the numerical values of G, M and R. the escape velocity is 11.2 km/s.

5.8 Earth Satellites:

The objects which revolve around the Earth are called Earth satellites. moon is the only natural satellite of the Earth. It revolves in almost a circular orbit around the Earth with period of revolution of nearly 27.3 days. Artificial satellites have been launched by several countries including India. These satellites have different periods of revolution according to their practical use like navigation, surveillance, communication, looking into space and monitoring the weather.

Communication Satellites: These are geostationary satellites. They revolve around the Earth in equatorial plane. They have same sense of rotation as that of the Earth and the same period of rotation as that of the Earth, i. e., one day or 24 hours. Due to this, they appear stationary from the Earth's surface. Hence they are called geostationary satellites or geosynchronous satellites. These are used for communication, television transmission, telephones and radiowave signal transmission, e.g., INSAT group of satellites launched by India.

Polar Satellites: These satellites are placed in lower polar orbits. They are at low altitude 500 km to 800 km. Polar satellites are used for weather forecasting and meteorological purpose. They are also used for astronomical observations and study of Solar radiations.

Period of revolution of polar satellite is nearly 85 minutes, so it can orbit the Earth16 time per day. They go around the poles of the Earth in a north-south direction while the Earth rotates in an east-west direction about its own axis. The polar satellites have cameras fixed on them. The camera can view small stripes of the Earth in one orbit. In entire day the whole Earth can be viewed strip by strip. Polar and equatorial regions at close distances can be viewed by these satellites.

5.8.1 Projection of Satellite:

For the projection of an artificial satellite, it is necessary for the satellite to have a certain velocity and a minimum two stage rocket. A single stage rocket can not achieve this. When the fuel in first stage of rocket is ignited on the surface of the Earth, it raises the satellite vertically. The velocity of projection of satellite normal to the surface of the Earth is the vertical velocity. If this vertical velocity is less that the escape velocity (v_e), the satellite returns to the Earth's surface. While, if the vertical velocity is greater than or equal to the escape velocity, the satellite will escape from Earth's gravitational influence and go to infinity. Hence launching of a satellite in an orbit round the Earth can not take place by use of single stage rocket. It requires minimum two stage rocket.

With the help of first stage of rocket, satellite can be taken to a desired height above the surface of the Earth. Then the launcher is rotated in horizontal direction i.e. through 90^0 using remote control and the first stage of the rocket is detached. Then with the help of second stage of rocket, a specific horizontal velocity (v_h) is given to satellite so that it can revolve in a circular path round the Earth. The exact horizontal velocity of projection that must be given to a satellite at a certain height so that it can revolve in a circular orbit round the Earth is called the **critical velocity** or **orbital velocity** (v_c)

A satellite follows different paths depending upon the horizontal velocity provided to it. Four different possible cases are shown in Fig. 5.10.

Case (I) $v_h<v_c$:

If tangential velocity of projection v_h is less than the critical velocity, the orbit of satellite is an ellipse with point of projection as apogee (farthest from the Earth) and Earth at one of the foci.

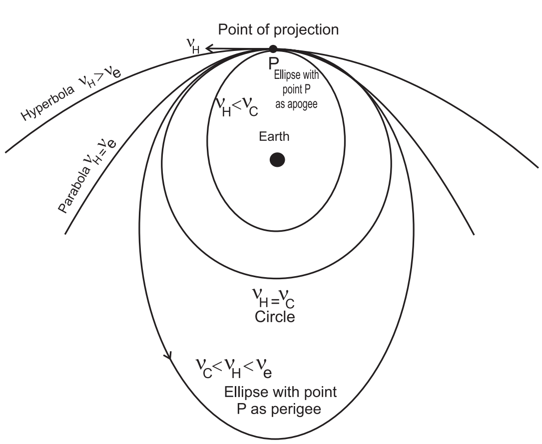


Fig. 5.10: Various possible orbits depending on the value of v_h.

During this elliptical path, if the satellite passes through the Earth's atmosphere, it experiences a nonconservative force of air resistance. As a result it loses energy and spirals down to the Earth.

Case (II) $v_h=v_c$

If the horizontal velocity is exactly equal to the critical velocity, the satellite moves in a stable circular orbit round the Earth.

Case (III) $v_c<v_h<v_e$

If horizontal velocity is greater than the critical velocity and less than the escape velocity at that height, the satellite again moves in an elliptical orbit round the Earth with the point of projection as perigee (point closest to the Earth).

Case (IV) $v_h = v_e$

If horizontal speed of projection is equal to the escape speed at that height, the satellite travels along parabolic path and never returns to the point of projection. Its speed will be zero at infinity.

Case (V) $v_h > v_e$

If horizontal velocity is greater than the escape velocity, the satellite escapes from gravitational influence of Earth transversing a hyperbolic path.

Expression for critical speed

Consider a satellite of mass m revolving round the Earth at height h above its surface.

Let M be the mass of the Earth and R be its radius. If the satellite is moving in a circular orbit of radius $(R+h) = r$, its speed must be the magnitude of critical velocity v_c.

The centripetal force necessary for circular motion of satellite is provided by gravitational force exerted by the Earth on the satellite.

$\therefore$ Centripetal force = Gravitational force

$$\frac{m v_c^{\,2}}{r} = \frac{GMm}{r^2}$$

$$\therefore v_c^{\,2} = \frac{GM}{r}$$

$$\therefore v_c = \sqrt{\frac{GM}{r}}$$

$$\therefore v_c = \sqrt{\frac{GM}{(R+h)}} = \sqrt{g_h (R+h)} \quad \text{--- (5.35)}$$

This is the expression for critical speed in the orbit of radius $(R + h)$

It is clear that the critical speed of a satellite is independent of the mass of the satellite. It depends upon the mass of the Earth and the height at which the satellite is revolving or gravitational acceleration at that altitude. The critical speed of a satellite decreases with increase in height of satellite.

Special case

When the satellite is revolving close to the surface of the Earth, the height is very small as compared to the radius of the Earth. Hence the height can be neglected and radius of the orbit is nearly equal to R (i.e $R >> h$, $R+h \approx R$)

$$\therefore \text{Critical speed } v_c = \sqrt{\frac{GM}{R}}$$

As G is related to acceleration due to gravity by the relation,

$$g = \frac{GM}{R^2}$$

$$\therefore GM = gR^2$$

$\therefore$ Critical speed in terms of acceleration due to gravity can be obtained as

$$v_c = \sqrt{\frac{gR^2}{R}} = \sqrt{gR}$$

$$= 7.92 \text{ km/s}$$

Obviously, this is the maximum possible critical speed. This is at least 25 times the speed of the fastest passenger aeroplanes.

Example 5.9: Show that the critical velocity of a body revolving in a circular orbit very close to the surface of a planet of radius R and mean density ρ is $2R\sqrt{\frac{G\pi\rho}{3}}$.

Solution : Since the body is revolving very close to the planet, $h = 0$

$$\text{density } \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\therefore M = \frac{4}{3}\pi R^3 \rho$$

Critical Velocity

$$v_c = \sqrt{\frac{GM}{R}} \quad = \sqrt{\frac{G\frac{4}{3}\pi R^3 \rho}{R}}$$

$$\therefore v_c = 2R\sqrt{\frac{G\pi\rho}{3}}$$

When a satellite revolves very close to the surface of the Earth, motion of satellite gets affected by the friction produced due to resistance of air. In deriving the above expression the resistance of air is not considered.

5.8.2 Weightlessness in a Satellite:

According to Newton's second law of motion, $F = ma$, where F is the net force acting on an object having acceleration a.

Let us consider the example of a lift or elevator from an inertial frame of reference.

Whether the lift is at rest or in motion, a passenger in it experiences only two forces: (i) Gravitational force mg directed vertically downwards (towards centre of the earth) and (ii) normal reaction force N directed vertically upwards, exerted by the floor of the lift. As these forces are oppositely directed, the net force in the downward direction will be $F = ma - N$.

Though the weight of a body (passenger, in this case) is the gravitational force acting upon it, we experience or *feel* our weight only due to the normal reaction force N exerted by the floor. This, in turn, is equal and opposite to the relative force between the body and the lift. If you are standing on a weighing machine in a lift, the force recorded by the weighing machine is nothing but the normal reaction N.

Case I: Lift having zero acceleration

This happens when the lift is at rest or is moving upwards or downwards with constant velocity:

The net force $F = 0 = mg - N \therefore mg = N$

Hence in this case we feel our normal weight mg .

Case II: Lift having net upward acceleration a_u

This happens when the lift just starts moving upwards or is about to stop at a lower floor during its downward motion (remember, while stopping during downward motion, the acceleration must be upwards).

As the net acceleration is upwards, the upward force must be greater.

$\therefore F = ma_u = N - mg \therefore N = mg + ma_u$, i.e., $N > mg$, hence, we *feel* heavier.

It should also be remembered that this is not an apparent feeling. The weighing machine really records a reading greater than *mg*.

Case III (a): Lift having net downward acceleration a_d

This happens when the lift just starts moving downwards or is about to stop at a higher floor during its upward motion (remember, while stopping during upward motion, the acceleration must be downwards).

As the net acceleration is downwards, the downward force must be greater.

$\therefore F = ma_d = mg - N \therefore N = mg - ma_d$, i.e., $N < mg$, hence, we *feel* lighter.

It should be remembered that this is not an apparent feeling. The weighing machine really records a reading less than *mg*.

Case III (b): State of free fall: This will be possible if the cables of the lift are cut. In this case, the downward acceleration $a_d = g$.

If the downward acceleration becomes equal to the gravitational acceleration *g*, we get, $N = mg - ma_d = 0$.

Thus, there will not be any feeling of weight. This is the state of *total* weightlessness and the weighing machine will record *zero*.

In the case of a revolving satellite, the satellite is performing a circular motion. The acceleration for this motion is centripetal, which is provided by the gravitational acceleration *g* at the location of the satellite. In this case, $a_d = g$, or the satellite (along with the astronaut) is in the state of free fall. Obviously, the apparent weight will be zero, giving the feeling of *total* weightlessness. Perhaps you might have seen in some videos that the astronauts are floating inside the satellite. It is really difficult for them to change their position.

In spite of free fall, why is the satellite not falling on the earth? The reason is that the revolving satellite is having a tangential velocity which manages to keep it moving in a circular orbit at that height.

5.8.3 Time Period of a Satellite:

The time taken by a satellite to complete one revolution round the Earth is its time period.

Consider a satellite of mass m projected to height *h* and provided horizontal velocity equal to the critical velocity. The satellite revolves in a circular orbit of radius $(R+h) = r$.

The distance traced by satellite in one revolution is equal to the circumference of the circular orbit within periodic time *T*.

$$\therefore \text{Critical speed} = \frac{\text{Circumference of the orbit}}{\text{Time period}}$$

$$v_c = \frac{2\pi r}{T}$$

but we have, $v_c = \sqrt{\dfrac{Gm}{r}}$

$$\therefore \sqrt{\frac{Gm}{r}} = \frac{2\pi r}{T}$$

or, $$\frac{GM}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$\therefore T^2 = \frac{4\pi^2 r^3}{GM}$$

As π^2, *G* and *M* are constant, $T^2 \propto r^3$, i.e., the square of period of revolution of satellite is directly proportional to the cube of the radius of orbit.

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\therefore T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} \quad \text{--- (5.36)}$$

This is an expression for period of satellite revolving in a *circular orbit* round the Earth. Period of a satellite does not depend on its mass. It depends on mass of the Earth, radius of the Earth and the height of the satellite. If the height of projection is increased, period of the satellite increases. Period of the satellite can also be obtained in terms of acceleration due to gravity.

As $GM = g_h (R+h)^2$

$$\therefore T = 2\pi\sqrt{\frac{(R+h)^3}{g_h (R+h)^2}}$$

$$\therefore T = 2\pi\sqrt{\frac{R+h}{g_h}}$$

$$\therefore T = 2\pi\sqrt{\frac{r}{g_h}} \qquad \text{--- (5.37)}$$

Special case :

When satellite revolves close to the surface of the Earth, $R + h \approx R$ and $g_h \approx g$. Hence the minimum period of revolution is

$$(T)_{min} = 2\pi\sqrt{\frac{R}{g}} \qquad \text{--- (5.38)}$$

Example 5.9: Calculate the period of revolution of a polar satellite orbiting close to the surface of the Earth. Given $R = 6400$ km, $g = 9.8$ m/s^2.

Solution : h is negligible as satellite is close to the Earth surface.

$$\therefore R + h \approx R$$

$$g_h \approx g$$

$$R = 6400 \text{ km} = 6.4\times10^6 \text{ m.}$$

$$T = 2\pi\sqrt{\frac{R}{g}}$$

$$= 2\times3.14\sqrt{\frac{6.4\times10^6}{9.8}}$$

$$= 5.075\times10^3 \text{ second}$$

$$= 85 \text{ minute (approximately)}$$

Example 5.10: An artificial satellite revolves around a planet in circular orbit close to its surface. Obtain the formula for period of the satellite in terms of density ρ and radius R of planet.

Solution : Period of satellite is given by,

$$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} \qquad \text{--- (1)}$$

Here, the satellite revolves close to the surface of planet, hence h is negligible, hence $R + h \simeq R$

$$\text{density } (\rho) = \frac{\text{mass } (M)}{\text{volume } (V)}$$

$$\therefore M = \rho V \qquad \text{--- (2)}$$

As planet is spherical in shape, volume of planet is given as

$$V = \frac{4}{3}\pi R^3$$

$$\therefore M = \frac{4}{3}\pi R^3 \rho \qquad \text{--- (3)}$$

Substituting the values form eq. (2) and (3) in Eq. (1), we get

$$T = 2\pi\sqrt{\frac{R^3}{G\times\frac{4}{3}\pi R^3 \rho}}$$

$$\therefore T = \sqrt{\frac{3\pi}{G\rho}}$$

5.8.4 Binding Energy of an orbiting satellite:

The minimum energy required by a satellite to escape from Earth's gravitational influence is the binding energy of the satellite.

Expression for Binding Energy of satellite revolving in circular orbit round the Earth

Consider a satellite of mass m revolving at height h above the surface of the Earth in a circular orbit. It possesses potential energy as well as kinetic energy. Let M be the mass of the Earth, R be the Radius of the Earth, v_c be critical velocity of satellite, $r = (R+h)$ be the radius of the orbit.

∴Kinetic energy of satellite

$$= \frac{1}{2} m v_c^2$$

$$= \frac{1}{2}\frac{GMm}{r} \qquad \text{--- (5.39)}$$

The gravitational potential at a distance r from the centre of the Earth is $-\dfrac{GM}{r}$

∴ Potential energy of satellite = Gravitational potential × mass of satellite

$$= -\frac{GMm}{r} \qquad \text{--- (5.40)}$$

The total energy of satellite is given as

$T.E. = K.E. + P.E.$

$$= \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

$$= -\frac{1}{2}\frac{GMm}{r} \quad \text{--- (5.41)}$$

Total energy of a circularly orbiting satellite is negative. Negative sign indicates that the satellite is bound to the Earth, due to gravitational force of attraction. For the satellite to be free from the Earth's gravitational influence its total energy should become non-negative (zero or positive). Hence the minimum energy to be supplied to unbind the satellite is $+\frac{1}{2}\frac{GMm}{r}$ This is the binding energy of a satellite.

Internet my friend

hyperphysics.phy-astr.gsu.edu/hbase/grav.html#grav

Exercises

1. Choose the correct option.

i) The value of acceleration due to gravity is maximum at ________

(A) the equator of the Earth .

(B) the centre of the Earth.

(C) the pole of the Earth.

(D) slightly above the surface of the Earth.

ii) The weight of a particle at the centre of the Earth is ________

(A) infinite.

(B) zero.

(C) same as that at other places.

(D) greater than at the poles.

iii) The gravitational potential due to the Earth is minimum at ________

(A) the centre of the Earth.

(B) the surface of the Earth.

(C) a points inside the Earth but not at its centre.

(D) infinite distance.

iv) The binding energy of a satellite revolving around planet in a circular orbit is 3×10^9 J. Its kinetic energy is ________

(A) 6×10^9J

(B) -3 $\times10^9$J

(C) -6 $\times10^{+9}$J

(D) 3 $\times10^{+9}$J

2. Answer the following questions.

i) State Kepler's law equal of area.

ii) State Kepler's law of period.

iii) What are the dimensions of the universal gravitational constant?

iv) Define binding energy of a satellite.

v) What do you mean by geostationary satellite?

vi) State Newton's law of gravitation.

vii) Define escape velocity of a satellite.

viii) What is the variation in acceleration due to gravity with altitude?

ix) On which factors does the escape speed of a body from the surface of Earth depend?

x) As we go from one planet to another planet, how will the mass and weight of a body change?

xi) What is periodic time of a geostationary satellite?

xii) State Newton's law of gravitation and express it in vector form.

xiii) What do you mean by gravitational constant? State its SI units.

xiv) Why is a minimum two stage rocket necessary for launching of a satellite?

xv) State the conditions for various possible orbits of a satellite depending upon the tangential speed of projection.

2. Answer the following questions in detail.

i) Derive an expression for critical velocity of a satellite.

ii) State any four applications of a communication satellite.

iii) Show that acceleration due to gravity at height h above the Earth's surface is $g_h = g\left(\frac{R}{R+h}\right)^2$

iv) Draw a labelled diagram to show different trajectories of a satellite depending upon the tangential projection speed.

v) Derive an expression for binding energy of a body at rest on the Earth's surface.

vi) Why do astronauts in an orbiting satellite have a feeling of weightlessness?

vii) Draw a graph showing the variation of gravitational acceleration due to the depth and altitude from the Earth's surface.

viii) At which place on the Earth's surface is the gravitational acceleration maximum? Why?

ix) At which place on the Earth surface the gravitational acceleration minimum? Why?

x) Define the binding energy of a satellite. Obtain an expression for binding energy of a satellite revolving around the Earth at certain attitude.

xi) Obtain the formula for acceleration due to gravity at the depth 'd' below the Earth's surface.

xii) State Kepler's three laws of planetary motion.

xiii) State the formula for acceleration due to gravity at depth 'd' and altitude 'h'

Hence show that their ratio is equal to $\left(\frac{R-d}{R-2h}\right)$ by assuming that the altitude is very small as compared to the radius of the Earth.

xiv) What is critical velocity? Obtain an expression for critical velocity of an orbiting satellite. On what factors does it depend?

xv) Define escape speed. Derive an expression for the escape speed of an object from the surface of the each.

xvi) Describe how an artificial satellite using two stage rocket is launched in an orbit around the Earth.

4. Solve the following problems.

i) At what distance below the surface of the Earth, the acceleration due to gravity decreases by 10% of its value at the surface, given radius of Earth is 6400 km.

[Ans: 640 km].

ii) If the Earth were made of wood, the mass of wooden Earth would have been 10% as much as it is now (without change in its diameter). Calculate escape speed from the surface of this Earth.

[Ans: 3.54 km/s]

iii) Calculate the kinetic energy, potential energy, total energy and binding energy of an artificial satellite of mass 2000 kg orbiting at a height of 3600 km above the surface of the Earth.

Given:- $G = 6.67\times10^{-11}$ Nm^2/kg^2

$R = 6400$ km

$M = 6\times10^{24}$ kg

[Ans: KE = 40.02×10^9J,

PE = -80.09×10^9J,

TE = -40.07×10^9J,

BE = 40.02×10^9J]

iv) Two satellites A and B are revolving around a planet. Their periods of revolution are 1 hour and 8 hours respectively. The radius of orbit of satellite B is 4×10^4 km. find radius of orbit of satellite A .

[Ans: 1×10^4 km]

v) Find the gravitational force between the Sun and the Earth.

Given Mass of the Sun = 1.99×10^{30} kg

Mass of the Earth = 5.98×10^{24} kg

The average distance between the Earth and the Sun = 1.5×10^{11} m.

[Ans: 3.5×10^{22} N]

vi) Calculate the acceleration due to gravity at a height of 300 km from the surface of the Earth. ($M = 5.98 \times10^{24}$ kg, $R = 6400$ km).

[Ans :- 8.889 m/s^2]

vii) Calculate the speed of a satellite in an orbit at a height of 1000 km from the Earth's surface. $M_E=5.98\times10^{24}$ kg, $R = 6.4\times10^6$ m.

[Ans : 7.34×10^3 m/s]

viii) Calculate the value of acceleration due to gravity on the surface of Mars if the radius of Mars = 3.4×10^3 km and its mass is 6.4×10^{23} kg.

[Ans : 3.69 m/s^2]

ix) A planet has mass 6.4×10^{24} kg and radius 3.4×10^6 m. Calculate energy required to remove on object of mass 800 kg from the surface of the planet to infinity.

[Ans : 5.02×10^{10}J]

x) Calculate the value of the universal gravitational constant from the given data. Mass of the Earth = 6×10^{24} kg, Radius of the Earth = 6400 km and the acceleration due to gravity on the surface = 9.8 m/s^2

[Ans : 6.69×10^{-11} N m^2/kg^2]

xi) A body weighs 5.6 kg wt on the surface of the Earth. How much will be its weight on a planet whose mass is 7 times the mass of the Earth and radius twice that of the Earth's radius.

[Ans: 9.8 kg-wt]

xii). What is the gravitational potential due to the Earth at a point which is at a height of $2R_E$ above the surface of the Earth, Mass of the Earth is 6×10^{24} kg, radius of the Earth = 6400 km and G = 6.67×10^{-11} Nm^2 kg^{-2}.

[Ans: 2.08×10^7 J]

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6. Mechanical Properties of Solids

Can you recall?

1. Can you name a few objects which change their shape and size on application of a force and regain their original shape and size when the force is removed ?
2. Can you name objects which do not regain their original shape and size when the external force is removed?

6.1 Introduction:

Solids are made up of atoms or a group of atoms placed in a definite geometric arrangement. This arrangement is decided by nature so that the resultant force acting on each constituent due to others is zero. This is the equilibrium state of a solid at room temperature. The given equilibrium arrangement does not change with time. It can change only when an external stimulus, like compressive force from all sides, is applied to a solid. The constituents vibrate about their equilibrium positions even at very low temperatures but cannot leave their fixed positions. This fact provides the solids a definite shape and size (allows the solids to maintain a definite shape and size).

If an external force is applied to a solid the constituents are slightly displaced and restoring forces are developed in it. These restoring forces try to bring the constituents back to their equilibrium positions so that the solid can regain its shape. When the deforming forces are removed, the interatomic forces tend to restore the original positions of the molecules and thus the body regains its original shape and size. However, as we will see later, this is possible only within certain limits.

The form of a body is decided by its size and shape, e.g., a tennis ball and a football both are spherical, i.e., they have the same shape. But a tennis ball is smaller in size than a football. When a force is applied to a solid (which is not free to move), the size or shape or both change due to changes in the relative positions of molecules. Such a force is called **deforming force**.

The change in shape or size or both of a body due to an external force is called deformation.

The larger the deforming force on a body, the larger is its deformation. Deformation could be in the form of change in length of a wire, change in volume of an object or change in shape of a body.

We know that when a deforming force (e.g. stretching) is applied to a rubber band, it gets deformed (elongated) but when the force is removed, it regains its original length. When a similar force is applied to a dough, or clay it also gets deformed but it does not regain its original shape and size after removal of the deforming force. These observations indicate that rubber and clay are different in nature. The property that decides this nature is called **elasticity/plasticity**. We will learn more about these properties of solids in this Chapter .

6.2 Elastic Behavior of Solids:

If a body regains its original shape and size after removal of the deforming force, it is called an elastic body and the property is called elasticity. Here the restoring forces are strong enough to bring the displaced molecules to their original positions. Examples of elastic materials are metals, rubber, quartz, etc.

If a body regains its original shape and size completely and instantaneously upon removal of the deforming force, then it is said to be **perfectly elastic.**

If a body does not regain its original shape and size and retains its altered shape or size upon removal of the deforming force, it is called a **plastic body** and the property is called **plasticity.** Here, the restoring forces are not strong enough to bring the molecules back to their original positions. Examples of plastic materials are clay, putty, plasticine, thick mud, etc. There is no solid which is perfectly elastic or perfectly plastic. The best example of a near ideal elastic solid is quartz fibre and that of a plastic body is putty.

6.3 Stress and Strain:

The elastic properties of a body are described in terms of stress and strain. When a body gets deformed under an applied force, restoring forces are set up internally. They oppose change in shape or size of the body. When body is in equilibrium in its altered shape or size, deforming force and restoring force are equal and opposite.

The internal restoring force per unit area of a body is called stress.

$$\text{stress} = \frac{\text{deforming force}}{\text{area}} = \frac{|\vec{F}|}{A} \quad \text{--- (6.1)}$$

where $\vec{F}$ is internal restoring force (external applied deforming force). SI unit of stress is N m^{-2} or pascal (Pa). The dimensions of a stress are [$L^{-1} M^{1} T^{-2}$].

Strain is a measure of the deformation of a body. When two equal and opposite forces are applied to an elastic body, there is a change in the dimensions of the body, **Strain is defined as the ratio of change in dimensions of the body to its original dimensions.**

$$\text{Strain} = \frac{\text{change in dimensions}}{\text{original dimensions}} \quad \text{--- (6.2)}$$

It is the ratio of two similar quantities. Hence strain is a dimensionless physical quantity. It has no units. There are three types of stress and corresponding strains.

1: Stress produced by a deforming force acting along the length of a body or a rod is called **tensile stress or a longitudinal stress**. The strain produced is called tensile strain.

A) Tensile stress or compressive stress:

Suppose a force $\vec{F}$ is applied along the length of a wire, or perpendicular to its cross section A. This produces an elongation in the wire and the length of the wire increases accordingly, as shown in Fig. 6.1 (a).

$$\text{Tensile stress} = \frac{|\vec{F}|}{A} \quad \text{--- (6.3)}$$

When a rod is pushed at two ends with equal and opposite forces, its length decreases. The restoring force per unit area is called **compressive stress** as shown in Fig. 6.1 (b).

$$\text{Compressive stress } \frac{|\vec{F}|}{A} \quad \text{--- (6.4)}$$

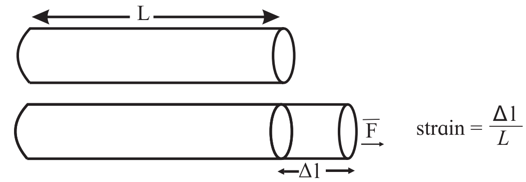


Fig. 6.1 (a): Tensile stress.

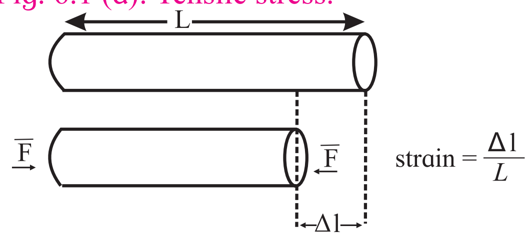


Fig. 6.1 (b): Compressive stress.

B) Tensile strain:

The strain produced by a tensile deforming force is called tensile strain or longitudinal strain or linear strain.

If L is the original length and Δl is the change in length due to the deforming force, then

$$\text{Tensile strain} = \frac{\Delta l}{L} \quad \text{--- (6.5)}$$

2 : When a deforming force acting on a body produces change in its volume, the stress is called volume stress and the strain produced is called **volume strain**.

A) Volume stress or hydraulic stress:

Let $\vec{F}$ be a force acting perpendicular to the entire surface of the body. It acts normally and uniformly all over the surface area A of the body. Such a stress which produces change in size but no change in shape is called volume stress.

$$\text{Volume stress} = \frac{|\vec{F}|}{A} \quad \text{--- (6.6)}$$

Volume stress produces change in size without change in shape of body, it is called hydraulic or hydrostatic volume stress as shown in Fig. 6.2.

B) Volume strain:

A deforming force acting perpendicular to the entire surface of a body produces a volume strain. Let V be the original volume and ΔV be the change in volume due to deforming force, then

$$\text{Volume stain} = \frac{\Delta V}{V} \quad \text{--- (6.7)}$$

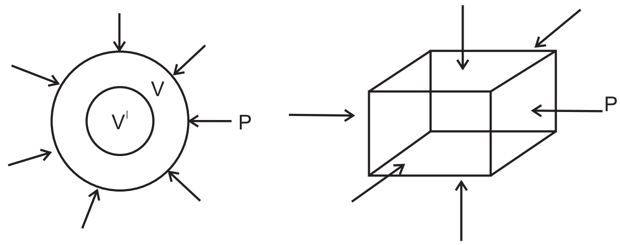


Fig. 6.2 : Volume stress.

Do you know ?

When a balloon is filled with air at high pressure, its walls experience a force from within. This is also volume stress. It tries to expand the balloon and change its size without changing shape. When the volume stress exceeds the limit of bulk elasticity, the balloon explodes. Similarly, a gas cylinder explodes when the pressure inside it exceeds the limit of bulk elasticity of its material.

A submarine when submerged under water is under volume stress.

3 : When a deforming force acting on a body produces change in the shape of a body, shearing stress and shearing strain are produced.

A) Shearing stress:

Let $\vec{F}$ be a tangential force acting on a surface area A. This force produces change in shape of the body without changing its size as shown in Fig. 6.3.

$$\text{Shearing stress} = \frac{\text{Tangential force}}{\text{Area}} \quad \text{--- (6.8)}$$

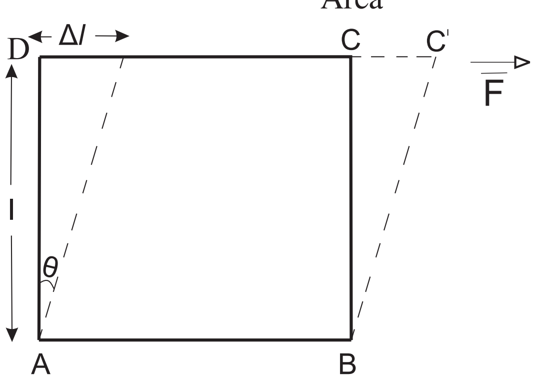


Fig. 6.3 : Tangential force produces shearing stress.

Suppose ABCD is the front face of a cube. A force $\vec{F}$ is applied to the cube so that the bottom of the cube is fixed and only the top surface is slightly displaced. Such force is called tangential force. **Tangential force is parallel to the top and the bottom surface of the block. The restoring force per unit area developed due to the applied tangential force is called shearing stress or tangential stress.**

B) Shearing strain:

There is a relative displacement, Δl, of the bottom face and the top face of the cube. Such relative displacement of two surfaces is called shear strain. It can be calculated as follows,

$$\text{Shearing strain } \frac{\Delta l}{l} = \tan\theta = \theta \quad \text{--- (6.9)}$$

when the relative displacement Δl is very small.

6.4 Hooke's Law:

Robert Hooke (1635-1703), an English physicist, studied the tension in a wire and strain produced in it. His study led to a law now known as Hooke's law.

Statement: Within elastic limit, stress is directly proportional to strain.

$$\frac{\text{Stress}}{\text{Strain}} = \text{constant}$$

The constant is called the modulus of elasticity. **The modulus of elasticity of a material is the ratio of stress to the corresponding strain.** It is defined as the slope of the stress-strain curve in the elastic deforming region and depends on the nature of the material. The maximum value of stress up to which stress is directly proportional to strain is called the **elastic limit**. The stress-strain curve within elastic limit is shown in Fig. 6.4

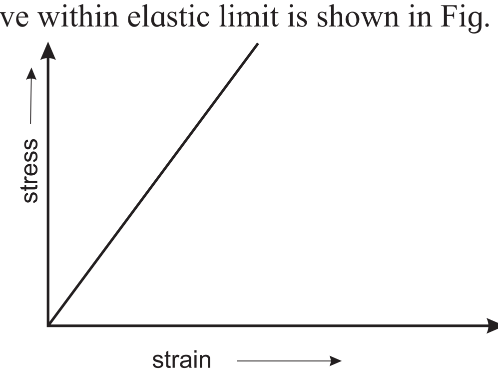


Fig 6.4: Stress versus strain graph within elastic limit for an elastic body.

6.5 Elastic modulus:

There are three types of stress and strain related to change in length, change in volume and change in shape. Hence, we have three moduli of elasticity corresponding to each type of stress and strain.

6.5.1 Young's modulus (Y):

It is the modulus of elasticity related to change in length of an object like a metal wire, rod, beam, etc., due to the applied deforming force. Hence it is also called as elasticity of length. It is named after the British physicist Thomas Young (1773-1829).

Consider a metal wire of length L having radius r suspended from a rigid support. A load Mg is attached to the free end of the wire. Due to this, deforming force is applied at the free end of the wire in downward direction. In its equilibrium position,

$$Longitudinal\ stress = \frac{Applied\ force}{Area}$$

$$= \frac{F}{A}$$

$$= \frac{Mg}{\pi r^2} \quad \text{--- (6.10)}$$

It produces a change in length of the wire. If (L+l) is the new length of wire, then l is the extension or elongation in wire.

$$Longitudinal\ strain = \frac{change\ in\ length}{original\ length}$$

$$= \frac{l}{L} \quad \text{--- (6.11)}$$

Young's modulus is the ratio of longitudinal stress to longitudinal strain.

$$Young's\ modulus = \frac{longitudinal\ stress}{longitudinal\ strain} \quad \text{-- (6.12)}$$

$$Y = \frac{\frac{Mg}{\pi r^2}}{\frac{l}{L}}$$

$$Y = \frac{MgL}{\pi r^2 l} \quad \text{---(6.13)}$$

SI unit of Young's modulus is N/m^2. Its dimensions are [$L^{-1} M^1 T^{-2}$].

Young's modulus indicates the resistance of an elastic solid to elongation or compression. Young's modulus of a material is useful for characterization of an object subjected to compression or tension. Young's modulus is the property of solids only.

Table 6.1: Young's modulus of some familiar materials

Material	Young's modulus Y $\times 10^{10}$ Pa (N/m^2)
Lead	1.5
Glass (crown)	6.0
Aluminium	7.0
Silver	7.6
Gold	8.1
Brass	9.0
Copper	11.0
Steel	21.0

Example 6.1: A brass wire of length 4.5m with crosssectional area of 3×10^{-5} m^2 and a copper wire of length 5.0 m with cross sectional area 4×10^{-5} m^2 are stretched by the same load. The same elongation is produced in both the wires. Find the ratio of Young's modulus of brass and copper.

Solution: For brass,

$L_B = 4.5$m, $A_B = 3\times10^{-5}$ m^2

$l_B = l$, $F_B = F$

$$Y_B = \frac{F_B\ L_B}{A_B\ l_B}$$

$$\therefore Y_B = \frac{F\times4.5}{3\times10^{-5}\times l}$$

For copper,

$L_C = 5$m, $A_C = 4\times10^{-5}$ m^2

$l_C = l$, $F_C = F$

$$Y_C = \frac{F_C\ L_C}{A_C\ l_C}$$

$$\therefore Y_c = \frac{F\times5.0}{4\times10^{-5}\times l}$$

$$\frac{Y_B}{Y_C} = \frac{F\times4.5}{3\times10^{-5}\times l}\times\frac{4\times10^{-5}\times l}{F\times5}$$

$$= \frac{18\times10^{-5}}{15\times10^{-5}} = 1.2$$

Example 6.2: A wire of length 20 m and area of cross section 1.25×10^{-4} m^2 is subjected to a load of 2.5 kg. (1 kgwt = 9.8 N). The elongation produced in wire is 1×10^{-4}m. Calculate Young's modulus of the material.

Solution: Given,

L = 20 m

A = 1.25×10^{-4} m^2

$F = mg = 2.5 \times 9.8N$

$L = 10^{-4}$ m

To find: Y

$$Y = \frac{FL}{Al} = \frac{2.5 \times 9.8 \times 20}{1.25 \times 10^{-4} \times 10^{-4}}$$

$$= 3.92 \times 10^{10} \text{ N m}^{-2}$$

6.5.2 Bulk modulus (K):

It is the modulus of elasticity related to change in volume of an object due to applied deforming force. Hence it is also called as elasticity of volume. Bulk modulus of elasticity is a property of solids, liquids and gases.

If a sphere made from rubber is completely immersed in a liquid, it will be uniformly compressed from all sides. Suppose this compressive force is F. Let the change in pressure on the sphere be dP and let the change in its volume be dV. If the original volume of the sphere is V, then volume strain is defined as

$$\text{Volume strain} = \frac{\text{change in volume}}{\text{original volume}}$$

$$= -\frac{dV}{V} \qquad \text{--- (6.14)}$$

The negative sign indicates that there is a decrease in volume. The magnitude of the volume strain is $\frac{dV}{V}$

Bulk modulus is defined as the ratio of volume stress to volume strain.

$$\text{Bulk modulus} = \frac{\text{volume stress}}{\text{volume strain}} = K$$

$$K = \frac{dP}{\left(\frac{dV}{V}\right)} = V\frac{dP}{dV} \qquad \text{--- (6.17)}$$

SI unit of bulk modulus is N/m^2. Dimensions of K are [$L^{-1} M^1 T^{-2}$].

Table 6.2 gives bulk moduli of some familiar materials

Bulk modulus measures the resistance offered by gases, liquids or solids while an attempt is made to change their volume.

The reciprocal of bulk modulus of elasticity is called compressibility of the material.

$$\text{Compressibility} = \frac{1}{\text{Bulk modulus}} \qquad \text{--- (6.18)}$$

Compressibility is the fractional decrease in volume, $-\Delta V/V$ per unit increase in pressure. SI unit of compressibility is m^2/ N or Pa^{-1} and its dimensions are [$L^1 M^{-1} T^2$].

Do you know ?

The bulk modules of water is 2.18×10^8 Pa and its compressibility is 45.8×10^{-10} Pa^{-1}. Materials with small bulk modulus and large compressibility are easier to compress.

Example 6.3: A metal cube of side 1m is subjected to a force. The force acts normally on the whole surface of cube and its volume changes by 1.5×10^{-5} m^3. The bulk modulus of metal is 6.6×10^{10} N/m^2. Calculate the change in pressure.

Solution: Given,

volume of cube=V = $l^3 = (1)^3 = 1\text{m}^3$

Change in volume = dV = 1.5×10^{-5} m^3

Bulk modulus = K = 6.6×10^{10} N/m^2.

To find: Change in pressure dP

$$K = V\frac{dP}{dV}$$

$$dP = K\frac{dV}{V}$$

$$dP = \frac{6.6\times10^{10}\times1.5\times10^{-5}}{1}$$

dP = 9.9×10^5 N/m^2.

Table 6.2: Bulk modulus of some familiar materials

Material	Bulk modulus K $\times10^{10}$ Pa (N/m^2)
Lead	4.1
Brass	6.0
Glass (crown)	6.0
Aluminium	7.5
Silver	10.0
Copper	14.0
Steel	16.0
Gold	18.0

6.5.3 Modulus of rigidity (η):

The modulus of elasticity related to change in shape of an object is called rigidity modulus. It is the property of solids only as they alone possess a definite shape.

The block shown in Fig. 6.5 is made of a uniform isotropic material. It has a uniform crosssection area A and height l. A cross section of the block is defined as any plane parallel to the top and the bottom surface and cuts the block. Two forces of magnitude 'F' are applied along top and bottom surface as shown in Fig. (6.5). They constitute a couple. The upper surface is displaced relative to the lower surface by a small distance Δl and corresponding angles change by a small amount $\theta = \Delta l/l$.

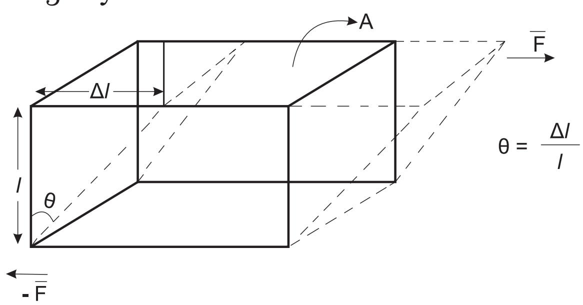


Fig. 6.5: Modulus of rigidity, tangential force F and shear strin θ.

A couple is applied by pushing the top and the bottom surfaces as shown in Fig. 6.5. Similar couple would be applied if the bottom of the block is fixed and only the top is pushed.

The forces $\vec{F}$ and -$\vec{F}$ are parallel to the cross section. This is different than the tensile stress where the force is normal to the cross section.

As a result of the way in which the forces are applied the block is subjected to a shear stress defined by **shear stress = *F/A***.

The SI unit of shear stress is N/m^2 or Pa. The block is distorted as a result of the shear stress. The top and bottom surface are relatively displaced by a small distance Δl. The corner angle changes by a small amount θ which is called shear strain and is expressed in radian. Shear strain 'θ' is given by $\theta = \Delta l/l$, (for small Δl).

Shear modulus or modulus of rigidity: It is defined as the ratio of shear stress to shear strain within elastic limits.

$$\eta = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\theta} = \frac{F}{A\theta} \quad \text{--- (6.17)}$$

Table 6.3 gives values of rigidity modulus η of some familiar materials.

Table 6.3: Rigidity modulus η of some familiar materials

Material	Rigidity modulus η $\times 10^{10}$ Pa (N/m^2)
Lead	0.6
Aluminium	2.5
Glass (crown)	2.5
Silver	2.7
Gold	2.9
Brass	3.5
Copper	4.4
Steel	8.3

Rigidity modulus indicates the resistance offered by a solid to change in its shape.

Example 6.4: Calculate the modulus of rigidity of a metal, if a metal cube of side 40 cm is subjected to a shearing force of 2000 N. The upper surface is displaced through 0.5cm with respect to the bottom. Calculate the modulus of rigidity of the metal.

Solution: Given,

Length of side of cube = l= 40 cm = 0.40 m

Shearing force = F= 2000 N = 2×10^3 N

Displacement of top face = Δl = 0.5cm = 0.005m

Area = $A = l^2 = 0{\cdot}16\text{m}^2$

To find: modulus of rigidity, η

$$\eta = \frac{F}{A\theta}$$

$$\theta = \frac{\Delta l}{l} = \frac{0.005}{0.40} = 0.0125$$

$$\eta = \frac{2.0\times10^3\,\text{N}}{(0.16\text{m}^2)\cdot(0.0125)}$$

$$= 1.0\times10^6\,\text{N/m}^2$$

6.5.4 Poisson's ratio:

Suppose a wire is fixed at one end and a force is applied at its free end so that the wire gets stretched. Length of the wire increases and at the same time, its diameter decreases, i.e., the wire becomes longer and thinner as shown in Fig. 6.6 (a).

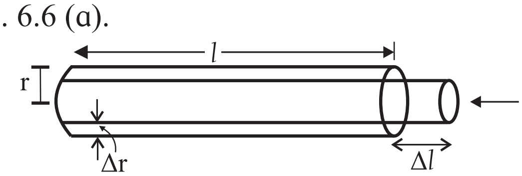


Fig. 6.6 (a): When a wire is stretched its length increases and its diameter decreases.

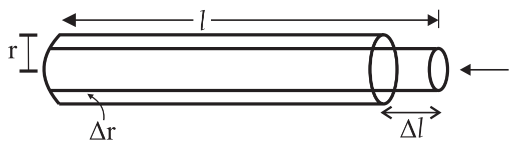


Fig. 6.6 (b): When a wire is compressed its length increases and its diameter increases.

If equal and opposite forces are applied to an object along its length inwards, the object gets compressed (Fig. 6.6 (b)). There is a decrease in dimensions along its length and at the same time there is an increase in its dimensions perpendicular to its length. When length of the wire decreases, its diameter increases.

The ratio of change in dimensions to original dimensions in the direction of the applied force is called **linear strain** while the ratio of change in dimensions to original dimensions in a direction perpendicular to the applied force is called **lateral strain.** Within elastic limit, the ratio of lateral strain to the linear strain is called the **Poisson's ratio**.

If l is the original length of wire, Δl is increase/decrease in length of wire, D is the original diameter and d is corresponding change in diameter of wire then, Poisson's ratio is given by

$$\sigma = \frac{Lateral\ strain}{Linear\ strain}$$

$$= \frac{d/D}{l/L}$$

$$= \frac{d.L}{D.l} \qquad \text{--- (6.18)}$$

Poisson's ratio has no unit. It is dimensionless. Table 6.4 gives values of Poisson ratio, σ, of some familiar materials.

Table 6.4: Poisson ratio, σ, of some familiar materials

Material	Poisson ratio σ
Glass (crown)	0.2
Steel	0.28
Aluminium	0.36
Brass	0.37
Copper	0.37
Silver	0.38
Gold	0.42

Do you know ?

For most of the commonly used metals, the value of σ is between 0.25 and 0.35. Many times we assume that volume is constant while stretching a wire. However, in reality, its volume also increases. Using approximations it can be shown that $\sigma_{max} \approx 0.5$ if volume is unchanged. In practice, it is much less. This shows that volume also increases while stretching.

6.6 Stress-Strain Curve:

Suppose a metal wire is suspended vertically from a rigid support and stretched by applying load to its lower end. The load is gradually increased in small steps until the wire breaks. The elongation produced in the wire is measured during each step. Stress and strain is noted for each load and a graph is drawn by taking tensile strain along x-axis and tensile stress along y-axis. It is a stress-strain curve as shown in Fig. 6.7.

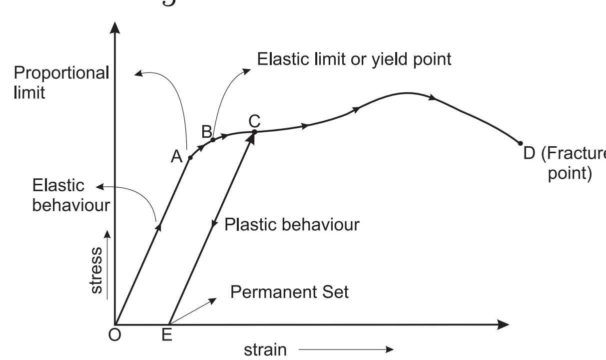


Fig. 6.7 : stress-strain curve.

The initial part of the graph is a straight line OA. This is the region in which Hooke's law is obeyed and stress is directly proportional to strain. The straight line portion ends at A. The stress at this point is called **proportional limit**. If the load is further increased till point B is reached, stress and strain are no longer proportional and Hooke's law is not valid. If the load is gradually removed starting at any point between O and B. The curve is retraced until the wire regains its original length. The change is reversible. The material of the wire shows elastic behaviour in the region OB. Point B is called the **yield point**. The corresponding point is called the **elastic limit**.

When the stress is increased beyond point B, the strain continues to increase. If the load is removed at any point beyond B, C for example, the material does not regain its original length. It follows the line CE. Length of the wire when there is no stress is greater than the original length. The deformation is irreversible and the material has acquired a **permanent set**.

Further increase in load causes a large increase in strain for relatively small increase in stress, until a point D is reached at which **fracture** takes place.

The material shows **plastic flow or plastic deformation** from point B to point D. The material does not regain its original state when the stress is removed. **The deformation is called plastic deformation.**

The curve described above shows all the possibilities for an elastic substance. In particular, many metallic wires (copper, aluminum, silver, etc) exhibit this type of behavior. However, majority of materials in every day life exhibit only some part of it.

Materials such as glass, ceramics, etc., break within the elastic limit. They are called **brittle**.

Metals such as copper, aluminum, wrought iron, etc. have large plastic range of extension. They lengthen considerably and undergo plastic deformation till they break. They are called **ductile**.

Metals such as gold, silver which can be hammered into thin sheets are called **malleable**.

Rubber has large elastic region. It can be stretched so that its length becomes many times its original length, after removal of the stress it returns to its original state but the stress strain curve is not a straight line. A material that can be elastically stretched to a larger value of strain is called an **elastomer**.

In case of some materials like vulcanized rubber, when the stress applied on a body decreases to zero, the strain does not return to zero immediately. The strain lags behind the stress. This lagging of strain behind the stress is called **elastic hysteresis**. Figure 6.8 shows the stress-strain curve for increasing and decreasing load. It encloses a loop. Area of loop gives the energy dissipated during deformation of a material.

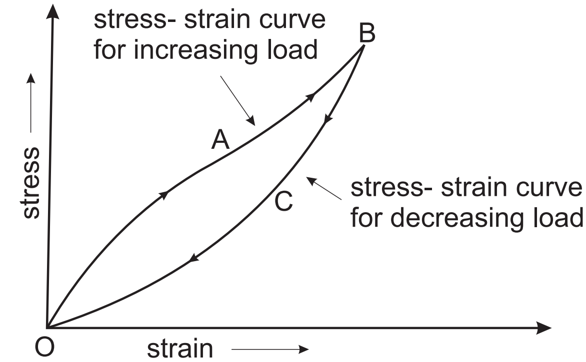


Fig. 6.8: Stress-stain curve for increasing and decreasing load.

Can you tell?

Why does a rubber band become loose after repeated use?

6.7 Strain Energy:

The elastic potential energy gained by a wire during elongation by a stretching force is called as strain energy.

Consider a wire of original length L and cross sectional area A stretched by a force F acting along its length. The wire gets stretched and elongation l is produced in it. The stress and the strain increase proportionately.

Longitudinal stress $= \dfrac{F}{A}$

Longitudinal strain $= \dfrac{l}{L}$

Young's modulus $= \dfrac{\text{longitudinal stress}}{\text{longitudinal strain}}$

$$Y = \frac{\left(\frac{F}{A}\right)}{\left(\frac{l}{L}\right)} = \frac{FL}{Al}$$

$$\therefore F = \frac{YAl}{L} \quad \text{--- (6.19)}$$

The magnitude of stretching force increases from zero to F during elongation of wire. At a certain stage, let 'f' be the force applied and 'x' be the corresponding extension. The force at this stage is given by Eq. (6.19) as

$$f = \frac{YAx}{L}$$

For further extension dx in the wire, the work done is given by

Work = (force)·(displacement).

dW = f dx

$$\therefore dW = \frac{YAx}{L}dx$$

When the wire gets stretched from x = 0 to x = l, the total work done is given as

$$W = \int_0^l dW$$

$$\therefore W = \int_0^l \frac{YAx}{L}dx$$

$$\therefore W = \frac{YA}{L}\int_0^l xdx$$

$$\therefore W = \frac{YA}{L}\left[\frac{x^2}{2}\right]_0^l$$

$$\therefore W = \frac{YA}{L}\left[\frac{l^2}{2} - \frac{0^2}{2}\right]$$

$$W = \frac{YAl^2}{2L}$$

$$W = \frac{1}{2}\frac{YAl}{L}l$$

$$W = \frac{1}{2}Fl$$

Work done = $\frac{1}{2}$ (load)·(extension) --- (6.20)

This work done by stretching force is equal to energy gained by the wire. This energy is strain energy.

Strain energy = $\frac{1}{2}$ (load)·(extension) --- (6.21)

Strain energy per unit volume can be obtained by using Eq. (6.20) and various formula of stress, strain and young's modulus.

Work done per unit volume

$$= \frac{work\ done\ in\ streching\ wire}{volume\ of\ wire.}$$

$$= \frac{1}{2}\frac{F.l}{A.L}$$

$$= \frac{1}{2}\left(\frac{F}{A}\right)\left(\frac{l}{L}\right)$$

Work done per unit volume

$$= \frac{1}{2}\ \text{(stress)·(strain)}$$

Strain energy per unit volume

$$= \frac{1}{2}\ \text{(stress)·(strain)} \quad \text{--- (6.22)}$$

As $Y = \frac{\text{stress}}{\text{strain}}$,

Stress = Y· (strain) and

strain $= \frac{\text{stress}}{\text{Y}}$

∴ Strain energy per unit volume

$$= \frac{1}{2}\text{Y}\cdot\text{(strain)}^2 \quad \text{--- (6.23)}$$

Also, strain energy per unit volume

$$= \frac{1}{2}\frac{\text{(stress)}^2}{\text{Y}} \quad \text{--- (6.24)}$$

Thus Eq. (6.22), (6.23) and (6.24) give strain energy per unit volume in various forms.

6.8 Hardness:

Hardness is the property of a material which enables it to resist plastic deformation. Hard materials have little ductility and they are brittle to some extent. **The term hardness also refers to stiffness or resistance to bending, scratching abrasion or cutting**. It is the property of a material which gives it the ability to resist permanent deformation when a load is applied to it. **The greater the hardness, greater is the resistance to deformation**.

The most well-known example of the hard materials is diamond. It is incredibly difficult to scratch a diamond. Metal with very low hardness is aluminium.

Hardness of material is different from its strength and toughness.

If a force is applied to a body it produces deformation in it. Higher is the force required for deformation, the stronger is the material, i.e., the material has more **strength**.

Steel has high strength whereas plasticine clay is not strong because it gets easily deformed even by a small force.

Toughness is the ability of a material to resist fracturing when a force is applied to it. Plasticine clay is relatively tough as it can be stretched and deformed due to applied force without breaking.

A single material may be hard, strong and tough, e.g.,

1) Bulletproof glass is hard and tough but not strong.
2) Drill bits must be hard, strong and tough for their work.
3) Anvils are very tough and strong but they are not hard.

6.9 Friction in Solids:

Whenever the surface of one body slides over another, each body exerts a certain amount of force on the other body. These forces are tangential to the surfaces. The force on each body is opposite to the direction of motion between the two bodies. It prevents or opposes the relative motion between the two bodies. It is a common experience that an object placed on any surface does not move easily when a small force is applied to it. This is because of certain force of opposition acting between the surface of the object and the surface on which it is placed. Even a rolling ball comes to rest after covering a finite distance on playground because of such opposing force. Our foot ware is provided with designs at the bottom of its sole so as to produce force of opposition to avoid slipping. It is difficult to walk without such opposing force. You know what happens when you try to walk fast on polished flooring at home with soap water spread on it. There is a possibility of slipping due to lack of force of opposition. To initiate any motion between a pair of surfaces, we need a certain minimum force. Also after the motion begins, it is constantly opposed by some natural force. This mechanical force between two solid surfaces in contact with each other is called as frictional force. **The property which resists the relative motion between two surfaces in contact is called friction.**

In some cases it is necessary to avoid friction, because friction causes dissipation of energy in machines due to which efficiency of machines decreases. In such cases friction should be reduced by using polished surfaces, lubricants, etc. Relative motion between solids and fluids (i.e. liquids and gases) is also naturally opposed by friction, e.g., a boat on the surface of water experiences opposition to its motion. In this section we are going to study friction in solids only.

6.9.1 Origin of friction:

If smooth surfaces are observed under powerful microscope, many irregularities and projections are observed. Friction arises due to interlocking of these irregularities between two surfaces in contact. The surfaces can be made extremely smooth by polishing to avoid irregularities but it is noticed that in this case also, friction does not decrease but may increase. Hence the interlocking of irregularities is not the real cause of friction.

According to modern theory, cause of friction is the force of attraction between molecules of two surfaces in actual contact in addition to the force due to the interlocking between the two surfaces. When one body is in contact with another body, the real microscopic area in contact is very small due to irregularities in contact. Figure 6.9 shows the microscopic view of two polished surfaces in contact.

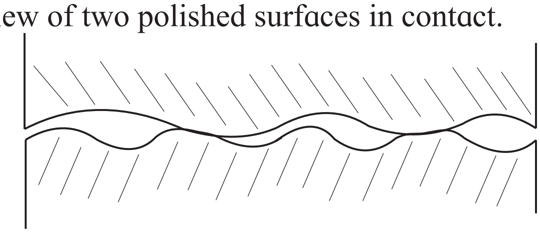


Fig. 6.9: Microscopic view of polished surfaces in contact.

Due to small area, pressure at points of contact is very high. Hence there is a strong force of attraction between the surfaces in contact. If both the surfaces are of the same material the force of attraction is called **cohesive force** while if the surfaces are of different materials the force of attraction is called **adhesive force**. When the surfaces in contact become more and more smooth, the actual area of contact goes on increasing. Due to this, the force of attraction between the molecules increases and hence the friction also increases. Putting some grease or other lubricant (a different material) between the two surfaces reduces the friction.

6.9.2 Types of friction:

1. Static friction:

Suppose a wooden block is placed on a horizontal surface as shown in Fig 6.10. A small horizontal force F is applied to it. The block does not move with this force as it cannot overcome the frictional force between the block and horizontal surface. In this case, the force of static friction is equal to F and balances it. **The frictional force which balances applied force when the body is static is called force of static friction, F_s. In other words, static friction prevents sliding motion.**

If we keep increasing $\underline{F}$, a stage will come when, for $\underline{F} = \underline{F}_{max}$, the object will start moving. For $\underline{F} < \underline{F}_{max}$, the force of static friction is equal to $\underline{F}$. For $\underline{F} \geq \underline{F}_{max}$, the kinetic friction comes into play. Static friction opposes impending motion i.e. the motion that would take place in absence of frictional force under the applied force.

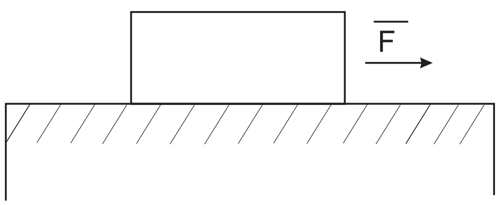


Fig. 6.10: Static friction.

The force of static friction is self adjusting force. When the applied force $\underline{F}$ is very small, the block remains at rest. Here the force of friction is also small. When $\underline{F}$ is increased by a small value, the block remains still at rest as the force of friction is increased to balance the applied force. If applied force is increased, the friction also increases and reaches the maximum value. Just before the body starts sliding over another body, the value of frictional force is maximum, it is called **limiting force of friction,** F_L. If the direction of applied force is reversed, the direction of static friction is also reversed, i.e., it adjusts its direction also.

Laws of static friction:

1] The limiting force of static friction is directly proportional to the normal reaction (N) between the two surfaces in contact.

$$F_L \propto N$$

$$\therefore F_L = \mu_s N \qquad \text{--- (6.25)}$$

Where μ_s is constant of proportionality. It is called as **coefficient of static friction.**

$$\therefore \mu_s = \frac{F_L}{N} \qquad \text{--- (6.26)}$$

The coefficient of static friction is defined as the ratio of limiting force of friction to the normal reaction. Table 6.4 gives the coefficient of static fiction for some materials.

2] The limiting force of friction is independent of the apparent area between the surfaces in contact, so long as the normal reaction remains the same.

3] The limiting force of friction depends upon materials in contact and the nature of their surfaces.

Table 6.4: Coefficient of static friction

Material	Coefficient of static friction μ_s
Teflon on Teflon	0.4
Brass on steel	0.51
Copper on steel	0.53
Aluminium on steel	0.61
Steel on steel	0.74
Glass on glass	0.94
Rubber on concrete (dry)	1.0

Example 6.5: The coefficient of static friction between a block of mass 0.25 kg and a horizontal surface is 0.4. Find the horizontal force applied to it.

Solution: Given,

$\mu_s = 0.4$

$m = 0.25$ kg

To find: Force

$$F = \mu_s \cdot N = \mu_s \cdot (mg)$$

$$F = 0.4 \times 0.25 \times 9.8$$

$$F = 0.98 \text{ N}$$

2. Kinetic friction :

Once the sliding of block on the surface starts, the force of friction decreases. The force required to keep the body sliding steadily is thus less than the force required to just start its sliding. The force of friction that comes into play when a body is in steady state of motion over another surface is called force of kinetic friction.

Friction between two surfaces in contact when one body is actually sliding over the other body, is called kinetic friction or dynamic friction.

Laws of kinetic friction :

1. The force of kinetic friction (F_k) is directly proportional to the normal reaction between two surfaces in contact.

   $\therefore F_k \propto N$

   $\therefore F_k = \mu_k N$ --- (6.27)

   Where μ_k is constant of proportionality. It is called as coefficient of kinetic friction.

   $$\therefore \mu_k = \frac{F_k}{N} \quad \text{--- (6.28)}$$

   The coefficient of kinetic friction is defined as the ratio of force of kinetic friction to the normal reaction between the two surfaces in contact. Table 6.5 gives the co-efficient of kinetic friction for some materials.

2. Force of kinetic friction is independent of shape and apparent area of the surfaces in contact.
3. Force of kinetic friction depends upon the nature and material of the surfaces in contact.
4. The magnitude of the force of kinetic friction is independent of the relative velocity between the object and the surface provided that the relative velocity is neither too large nor too small.

Table 6.5: Coefficient of kinetic friction

Material	Coefficient of kinetic friction μ_k
Rubber on concrete (dry)	0.25
Glass on glass	0.40
Brass on steel	0.40
Copper on steel	0.44
Aluminium on steel	0.47
Steel on steel	0.57
Teflon on Teflon	0.80

3 Rolling friction :

Motion of a body over a surface is said to be rolling motion if the point of contact of the body with the surface keeps changing continuously.

Friction between two bodies in contact when one body is rolling over the other, is called rolling friction.

For same pair of surfaces, the force of static friction is greater than the force of kinetic friction while the force of kinetic friction is greater than force of rolling friction. As rolling friction is the minimum, ball bearings are used to reduce friction in parts of machines to increase its efficiency.

Advantages of friction:

Friction is necessary in our daily life.

- We can walk due to friction between ground and feet.
- We can hold object in hand due to static friction.
- Brakes of vehicles work due to friction; hence we can reduce speed or stop vehicles.
- Climbing on a tree is possible due to friction.

Disadvantages of friction

- Friction opposes motion.
- Friction produces heat in different parts of machines. It also produces noise.
- Automobile engines consume more fuel due to friction.

Methods of reducing friction

- Use of lubricants, oil and grease in different parts of a machine.
- Use of ball bearings converts kinetic friction into rolling friction.

Can you tell?

1) It is difficult to run fast on sand.
2) It is easy to roll than pull a barrel along a road.
3) An inflated tyre rolls easily than a flat tyre.
4) Friction is a necessary evil.

Internet my friend

1. https://opentextbc.ca>chapter>friction.
2. https://www.livescience.com
3. https://www.khanacdemy.org.physics
4. https://courses.lumenlearning.com>elastiscitychapter>elasticity
5. https://www.toper.com>guides>physics

Exercises

1. Choose the correct answer:

i) Change in dimensions is known as…..

(A) deformation (B) formation
(C) contraction (D) strain.

ii) The point on stress-strain curve at which strain begins to increase even without increase in stress is called….

(A) elastic point (B) yield point
(C) breaking point (D) neck point

iii) Strain energy of a stretched wire is 18×10^{-3} J and strain energy per unit volume of the same wire and same cross section is 6×10^{-3} J/m^3. Its volume will be....

(A) 3cm^3 (B) 3 m^3
(C) 6 m^3 (D) 6 cm^3

iv) ----- is the property of a material which enables it to resist plastic deformation.

(A) elasticity (B) plasticity
(C) hardness (D) ductility

v) The ability of a material to resist fracturing when a force is applied to it, is called……

(A) toughness (B) hardness
(C) elasticity (D) plasticity.

2. Answer in one sentence:

i) Define elasticity.
ii) What do you mean by deformation?
iii) State the SI unit and dimensions of stress.
iv) Define strain.
v) What is Young's modulus of a rigid body?
vi) Why bridges are unsafe after a very long use?
vii) How should be a force applied on a body to produce shearing stress?
viii) State the conditions under which Hooke's law holds good.
ix) Define Poisson's ratio.
x) What is an elastomer?
xi) What do you mean by elastic hysteresis?
xii) State the names of the hardest material and the softest material.
xiii) Define friction.
xiv) Why force of static friction is known as 'self-adjusting force'?
xv) Name two factors on which the co-efficient of friction depends.

3. Answer in short:

i) Distinguish between elasticity and plasticity.
ii) State any four methods to reduce friction.
iii) What is rolling friction? How does it arise?
iv) Explain how lubricants help in reducing friction?
v) State the laws of static friction.
vi) State the laws of kinetic friction.
vii) State advantages of friction.
viii) State disadvantages of friction.
ix) What do you mean by a brittle substance? Give any two examples.

4. Long answer type questions:

i) Distinguish between Young's modulus, bulk modulus and modulus of rigidity.
ii) Define stress and strain. What are their different types?
iii) What is Young's modulus? Describe an experiment to find out Young's modulus of material in the form of a long straight wire.
iv) Derive an expression for strain energy per unit volume of the material of a wire.
v) What is friction? Define coefficient of static friction and coefficient of kinetic friction. Give the necessary formula for each.
vi) State Hooke's law. Draw a labeled graph of tensile stress against tensile strain for a metal wire up to the breaking point. In this graph show the region in which Hooke's law is obeyed.

5. Answer the following

i) Calculate the coefficient of static friction for an object of mass 50 kg placed on horizontal table pulled by attaching a spring balance. The force is increased gradually it is observed that the object just moves when spring balance shows 50N.

[Ans: $\mu_s = 0.102$]

ii) A block of mass 37 kg rests on a rough horizontal plane having coefficient of static friction 0.3. Find out the least force required to just move the block horizontally.

[Ans: $F_s = 108.8$N]

iii) A body of mass 37 kg rests on a rough horizontal surface. The minimum horizontal force required to just start the motion is 68.5 N. In order to keep the body moving with constant velocity, a force of 43 N is needed. What is the value of a) coefficient of static friction? and b) coefficient of kinetic friction?

[Ans: a) $\mu_s = 0.188$
b) $\mu_k = 0.118$]

iv) A wire gets stretched by 4mm due to a certain load. If the same load is applied to a wire of same material with half the length and double the diameter of the first wire. What will be the change in its length?

[Ans: 0.5mm]

v) Calculate the work done in stretching a steel wire of length 2m and cross sectional area 0.0225mm^2 when a load of 100 N is slowly applied to its free end. [Young's modulus of steel= 2×10^{11} N/m^2]

[Ans: 2.222J]

vi) A solid metal sphere of volume 0.31m^3 is dropped in an ocean where water pressure is 2×10^7 N/m^2. Calculate change in volume of the sphere if bulk modulus of the metal is 6.1×10^{10} N/m^2

[Ans: 10^{-4} m^3]

vii) A wire of mild steel has initial length 1.5 m and diameter 0.60 mm is extended by 6.3 mm when a certain force is applied to it. If Young's modulus of mild steel is 2.1 x 10^{11} N/m^2, calculate the force applied.

[Ans: 250 N]

viii) A composite wire is prepared by joining a tungsten wire and steel wire end to end. Both the wires are of the same length and the same area of cross section. If this composite wire is suspended to a rigid support and a force is applied to its free end, it gets extended by 3.25mm. Calculate the increase in length of tungsten wire and steel wire separately.

[Given: $Y_{steel} = 2 \times 10^{11}$N/m^2,
$Y_{Tungsten} = 3.40 \times 10^8$ N/m^2]

[Ans: extension in tungsten wire = 3.244 mm,
extension in steel wire = 0.0052 mm]

ix) A steel wire having cross sectional area 1.2 mm^2 is stretched by a force of 120 N. If a lateral strain of 1.455 mm is produced in the wire, calculate the Poisson's ratio.

[Given: $Y_{steel} = 2 \times 10^{11}$N/m^2]

[Ans: 0.291]

x) A telephone wire 125m long and 1mm in radius is stretched to a length 125.25m when a force of 800N is applied. What is the value of Young's modulus for material of wire?

[Ans: 1.27×10^{11}N/m^2]

xi) A rubber band originally 30cm long is stretched to a length of 32cm by certain load. What is the strain produced?

[Ans: 6.667×10^{-2}]

xii) What is the stress in a wire which is 50m long and 0.01cm^2 in cross section, if the wire bears a load of 100kg?

[Ans: 9.8×10^8 N/m^2]

xiii) What is the strain in a cable of original length 50m whose length increases by 2.5cm when a load is lifted?

[Ans: 5×10^{-4}]

***

7. Thermal Properties of Matter

Can you recall?

1. Temperature of a body determines its hotness while heat energy is its heat content.
2. Pressure is the force exerted per unit area normally on the walls of a container by the gas molecules due to collisions.
3. Solids, liquids and gases expand on heating.
4. Substances change their state from solid to liquid or liquid to gas on heating up to specific temperature.

7.1 Introduction:

In previous lessons, while describing the equilibrium states of a mechanical system or while studying the motion of bodies, only three fundamental physical quantities namely length, mass and time were required. All other physical quantities in mechanics or related to mechanical properties can be expressed in terms of these three fundamental quantities. In this chapter, we will discuss properties or phenomena related to heat. These require a fourth fundamental quantity, the temperature, as mentioned in Chapter 1.

The sensation of hot or cold is a matter of daily experience. A mother feels the temperature of her child by touching its forehead. A cook throws few drops of water on a frying pan to know if it is hot enough to spread the *d*osa batter. Although not advisable, in our daily lives, we feel hotness or coldness of a body by touching or we dip our fingers in water to check if it is hot enough for taking bath. When we say a body or water is hot, we actually mean that its temperature is more than our hand. However, in this way, we can only compare the hotness or coldness of two objects **qualitatively**. Hot and cold are relative terms. You might recall the example given in your science textbook of VIIIth standard. Lukewarm water seems colder than hot water but hotter than cold water to our hands. We ascribe a property 'temperature' to an object to determine its degree of hotness. The higher the temperature, the hotter is the body. However, the precise temperature of a body can be known only when we have an accurate and easily reproducible way to quantitatively measure it. Scientific precision requires measurement of a physical quantity in numerical terms. A thermometer is the device to measure the temperature.

In this chapter , we will learn properties of matter and various phenomena that are related to heat. Phenomena or properties having to do with temperature changes and heat exchanges are termed as thermal phenomena or thermal properties. You will understand why the direction of wind near a sea shore changes during day and night, why the metal lid of a glass bottle comes out easily on heating and why two metal vessels locked together can be separated by providing heat to the outer vessel.

7.2 Temperature and Heat:

Heat is energy in transit. When two bodies at different temperatures are brought in contact, they exchange heat. After some time, the heat transfer stops and we say the two bodies are in thermal equilibrium. The property or the deciding factor to determine the state of thermal equilibrium is the temperature of the two bodies. Temperature is a physical quantity that defines the thermodynamic state of a system.

You might have experienced that a glass of ice-cold water when left on a table eventually warms up whereas a cup of hot tea on the same table cools down. It means that when the temperature of a body, ice-cold water or hot tea in the above examples, is different from its surrounding medium, heat transfer takes place between the body and the surrounding medium until the body and the surrounding medium are at the same temperature. We then say that the body and its surroundings have reached a state of thermal equilibrium and there is no net transfer of heat from one to the other. In fact, whenever two bodies are in contact, there is a transfer of heat owing to their temperature difference.

Matter in any state - solid, liquid or gas- consists of particles (ions, atoms or molecules). In solids, these particles are vibrating about their fixed equilibrium positions and possess kinetic energy due to motion at the given temperature. The particles possess potential energy due to the interatomic forces that hold the particles together at some mean fixed positions. Solids therefore have definite volume and shape. When we heat a solid, we provide energy to the solid. The particles then vibrate with higher energy and we can see that the temperature of the solid increases (except near its melting point). Thus the energy supplied to the solid (does not disappear!) becomes the internal energy in the form of increased kinetic energy of atoms/ molecules and raises the temperature of the solid. The temperature is therefore a measure of the average kinetic energy of the atoms/ molecules of the body. The greater the kinetic energy is, the faster the molecules will move and higher will be the temperature of the body. If we continue heating till the solid starts to melt, the heat supplied is used to weaken the bonds between the constituent particles. The average kinetic energy of the constituent particles does not change further. The order of magnitude of the average distance between the molecules of the melt remains almost the same as that of solid. Due to weakened bonds liquids do not possess definite shape but have definite volume. The mean distance between the particles and hence the density of liquid is more or less the same as that of the solid. On heating further, the atoms/molecules in liquid gain kinetic energy and temperature of the liquid increases. If we continue heating the liquid further, at the boiling point, the constituents can move freely overcoming the interatomic/molecular forces and the mean distance between the constituents increases so that the particles are farther apart.

As per kinetic theory of gases, for an ideal gas, there are no forces between the molecules of a gas. Hence gases neither have a definite volume nor shape. Interatomic spacing in solids is ~ 10^{-10} m while the average spacing in liquids is almost twice that in solids. The average inter molecular spacing in gases at normal temperature and pressure (NTP) is ~10^{-9} m.

From the above discussion, we understand that **heat supplied to the substance increases the kinetic energy of molecules or atoms of the substance. The average kinetic energy per particle of a substance defines the temperature.** Temperature measures the degree of hotness of an object and not the amount of its thermal energy.

A glass of water, a gas enclosed in a container, a block of copper metal are all examples of a 'system'. We can say that heat in the form of energy is transferred between two (or more) systems or a system and its surroundings by virtue of their temperature difference. SI unit of heat energy is joule (J) and that of temperature is kelvin (K) or celcius (°C). The CGS unit of heat energy is erg. (1J = 10^7 erg). The other unit of heat energy, that you have learnt in VIIIth standard, is calorie (cal) and the relation with J is 1 cal = 4.184 J. Heat being energy has dimension $[L^2M^1T^{-2}K^0]$ while dimension of temperature is $[L^0M^0T^0K^1]$.

7.3 Measurement of Temperature:

In order to isolate two liquids or gases from each other and from the surroundings, we use containers and partitions made of materials like wood, plastic, glass wool, etc. An ideal wall or partition (not available in practice) separating two systems is one that does not allow any flow or exchange of heat energy from one system to the other. Such a perfect thermal insulator is called an **adiabatic wall** and is generally shown as a **thick cross-shaded** (slanting lines) region. When we wish to allow exchange of heat energy between two systems, we use a partition like a thin sheet of copper. It is termed as a **diathermic** wall and is represented as a thin dark region.

Let us consider two sections of a container separated by an adiabatic wall. Let them contain two different gases. Let us call them system A and system B. We independently bring systems A and B in thermal equilibrium with a system C. Now if we remove the adiabatic wall separating systems A and B, there will be no transfer of heat from system A to system B or vice versa. This indicates that systems A and B are also in thermal equilibrium. Overall conclusion of this activity can be summarized as follows: If systems A and B are separately in thermal equilibrium with a system C, then A and B are also mutually in thermal equilibrium. When two or more systems/ bodies are in thermal equilibrium, their temperatures are same. This principle is used to measure the temperature of a system by using a thermometer.

Do you know ?

If $T_A = T_B$ and $T_B = T_C$, then $T_A = T_C$ is not a mathematical statement, if T_X represents the temperature of system X. It is the zeroth law of thermodynamics and makes the science of Thermometry possible.

Do you remember that to know the temperature of our body, doctor brings the mercury in the thermometer down to indicate some low temperature. We are then asked to keep the thermometer in our mouth. We have to wait for some time before the thermometer is taken out to know the temperature of our body. There is transfer of heat energy from our body to the thermometer since initially our body is at a higher temperature. When the temperature on the thermometer is same as that of our body, thermal equilibrium is said to be attained and heat transfer stops.

As mentioned above, to precisely know the thermodynamic state of any system, we need to know its temperature. The device used to measure temperature is a thermometer. Thermometry is the science of temperature and its measurement. For measurement of temperature, we need to establish a temperature scale and adopt a set of rules for assigning numbers (with corresponding units).

For the calibration of a thermometer, a standard temperature interval is selected between two easily reproducible fixed temperatures just as we select the standard of length (metre) to be the distance between two fixed marks. The fact that substances change state from solid to liquid to gas at fixed temperatures is used to define reference temperature called fixed point. The two fixed temperatures selected for this purpose are the melting point of ice or freezing point of water and the boiling point of water. The next step is to sub-divide this standard temperature interval into sub-intervals by utilizing some physical property that changes with temperature and call each sub-interval a degree of temperature. This procedure sets up an empirical scale for temperature.

* The temperature at which pure water freezes at one standard atmospheric pressure is called **ice point**/ freezing point of water. This is also the melting point of ice.
* The temperature at which pure water boils and vaporizes into steam at one standard atmospheric pressure is called **steam point/** boiling point. This is also the temperature at which steam changes to liquid water.

Having decided the fixed point phenomena, it remains to assign numerical values to these fixed points and the number of divisions between them. In 1750, conventions were adopted to assign (i) a temperature at which pure ice melts at one atmosphere pressure (the ice point) to be 0º and (ii) a temperature at which pure water boils at one atmosphere (the steam point) to be 100º so that there are 100 degrees between the fixed points. This was the centigrade scale (*centi* meaning hundred in Latin). This was redefined as celcius scale after the Swedish scientist Anders Celcius (1701-1744). It is a convention to express temperature as degree celcius (ºC).

To measure temperature quantitatively, generally two different scales of temperature are used. They are describe below.

1) **Celsius scale:-** On this scale, the ice point is marked as 0 and the steam point is marked as 100, both taken at normal atmospheric pressure (10^5 Pa or N/m^2). The interval between these points is divided into 100

equal parts. Each of these is known as degree Celsius and is written as ºC.

2) **Fahrenheit scale** :- On this scale, the ice point is market as 32 and the steam point is marked as 212, both taken at normal atmospheric pressure. The interval between these points is divided into 180 equal parts. Each division is known as degree Fahrenheit and is written as °F.

A relationship for conversion between the two scales may be obtained from a graph of fahrenheit temperature (T_F) versus celsius temperature (T_C). The graph is a straight line (Fig. 7.1) whose equation is

$$\frac{T_F - 32}{180} = \frac{T_C}{100} \qquad \text{--- (7.1)}$$

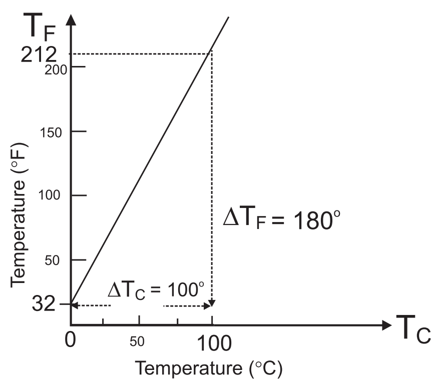


Fig. 7.1: A plot of fahrenheit temperature (T_F) versus celsius temperature (T_C).

Example 7.1: Average room temperature on a normal day is 27 °C. What is the room temperature in °F?

Solution: We have

$$\frac{T_F - 32}{180} = \frac{T_C}{100}$$

$$\therefore T_F = \frac{180}{100} T_C + 32$$

Given $T_C = 27$ °C,

$$T_F = \frac{180}{100} \times 27 + 32$$

$$= 48.6 + 32$$

$$= 80.6 \text{ °F}$$

Example 7.2: Normal human body temperature in feherenheit is 98.4 °F. What is the body temperature in °C?

Solution: We have

$$\frac{T_C}{100} = \frac{T_F - 32}{180}$$

$$\therefore T_C = \frac{100}{180} (T_F\text{-}32)$$

Given $T_F = 98.4$ °F,

$$T_C = \frac{100}{180}(98.4\text{-}32)$$

$$= \frac{100}{180}(66.4)$$

$$= 36.89 \text{ °C}$$

A device used to measure temperature, is based on the principle of thermal equilibrium. To measure the temperature, we use different measurable properties of materials which change with temperature. Some of them are length of a rod, volume of a liquid, electrical resistance of a metal wire, pressure of a gas at constant volume etc. Such changes in physical properties with temperature are used to design a thermometer. Physical property that is used in the thermometer for measuring the temperature is called the thermometric property and the material employed for the purpose is termed as the thermometric substance. Temperature is measured by exploiting the continuous monotonic variation of the chosen property with temperature. A calibration, however, is required to define the temperature scale.

There are different kinds of thermometers each type being more suitable than others for a certain job. In each type, the physical property used to measure the temperature must vary continuously over a wide range of temperature. It must be accurately measurable with simple apparatus.

An important characteristic of a thermometer is its *sensitivity*, i.e., a change in the thermometric property for a very small change in temperature. Two other characteristics are *accuracy* and *reproducibility*. Also it is important that the system attains thermal equilibrium with the thermometer quickly.

If the values of a thermometric property are P_1 and P_2 at the ice point (0 ºC) and steam point (100 ºC) respectively and the value of this property is P_T at unknown temperature T, then T is given by the following equation

$$T = \frac{100\left(P_T - P_1\right)}{\left(P_2 - P_1\right)} \qquad \text{--- (7.2)}$$

Ideally, there should be no difference

in temperatures recorded on two different thermometers. This is seen for thermometers based on gases as thermometric substances. In a constant volume gas thermometer, the pressure of a fixed volume of gas (measured by the difference in height) is used as the thermometric property. It is an accurate but bulky instrument.

Liquid-in-glass thermometer depends on the change in volume of the liquid with temperature. The liquid in a glass bulb expands up a capillary tube when the bulb is heated. The liquid must be easily seen and must expand (or contract) rapidly and by a large amount for a small change in temperature over a wide range of temperature. Most commonly used liquids are mercury and alcohol as they remain in liquid state over a wide range. Mercury freezes at -39 °C and boils at 357 °C; alcohol freezes at -115 °C and boils at 78 °C. Thermochromic liquids are ones which change colour with temperature but have a limited range around room temperatures. For example, titanium dioxide and zinc oxide are white at room temperature but when heated change to yellow.

Example 7.3: The length of a mercury column in a mercury-in-glass thermometer is 25 mm at the ice point and 180 mm at the steam point. What is the temperature when the length is 60 mm?

Solution: Here the thermometric property P is the length of the mercury column. Using Eq. (7.2), we get

$$T = \frac{100(60-25)}{(180-25)} = 22.58\,°\mathrm{C}$$

Resistance thermometer uses the change of electrical resistance of a metal wire with temperature. It measures temperature accurately in the range -2000 °C to 1200 °C but it is bulky and is best for steady temperatures.

Example 7.4: A resistance thermometer has resistance 95.2 Ω at the ice point and 138.6 Ω at the steam point. What resistance would be obtained if the actual temperature is 27 ºC?

Solution: Here the thermometric property P is the resistance. Using Eq. (7.2), if R is the resistance at 27 ºC, we have

$$27 = \frac{100(R-95.2)}{(138.6-95.2)},$$

$$\therefore R = \frac{27\times(138.6-95.2)}{100} + 95.2$$

$$= 11.72 + 95.2 = 106.92\ \Omega$$

Normally in research laboratories, a thermocouple is used to measure the temperature. A thermocouple is a junction of two different metals or alloys e.g., copper and iron joined together. When two such junctions at the two ends of two dissimilar metal rods are kept at two different temperatures, an electromotive force is generated that can be calibrated to measure the temperature.

Thermistor is another device used to measure temperature based on the change in resistance of a semiconductor materials i.e., the resistance is the thermometric property. You will learn more about this device in Chapter 14 on Semiconductors.

7.4 Absolute Temperature and Ideal Gas Equation:

7.4.1 Absolute zero and absolute temperature

Experiments carried out with gases at low densities indicate that while pressure is held constant, the volume of a given quantity of gas is directly proportional to temperature (measured in ºC). Similarly, if the volume of a given quantity of gas is held constant, the pressure of the gas is directly proportional to temperature (measured in ºC). These relations are graphically shown in Fig. 7.2 (a) and (b). Mathematically, this relationship can be written as $PV \propto T_C$. Thus the volume-temperature or pressure-temperature graphs for a gas are straight lines. They show that gases expand linearly with temperature on a mercury thermometer i.e., equal temperature increase causes equal volume or pressure increase. The similar thermal behavior of all gases suggests that this relationship of gases can be used to measure temperature in a constant-volume gas thermometer in terms of pressure of the gas.

Although actual experimental measurements might differ a little from the ideal linear relationship, the linear relationship holds over a wide temperature range.

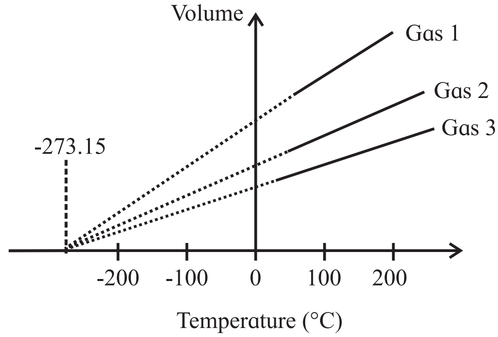


Fig. 7.2 (a): Graph of volume versus temperature (in °C) at constant pressure.

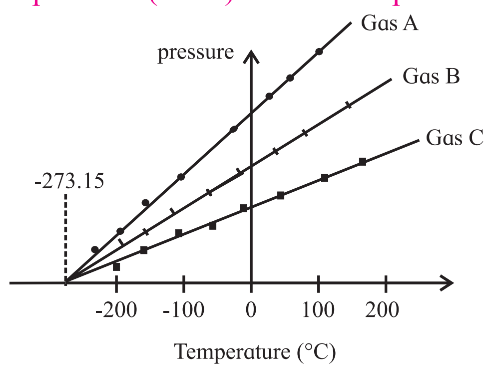


Fig. 7.2 (b): Graph of pressure versus temperature (in °C) at constant volume.

It may be noted that the lines do not pass through the origin i.e., have non-zero intercept along the *y*-axis. The straight lines have different slopes for different gases. If we assume that the gases do not liquefy even if we lower the temperature, we can extend the straight lines backwards for low temperatures. Is it possible to reach a temperature where the gases stop exerting any pressure i.e., pressure is zero? In a constant pressure thermometer, as the temperature is lowered, the volume decreases. Suppose the gas does not liquefy even at very low temperature, at what temperature, will its volume become zero? Practically it is not possible to keep a material in gaseous state for very low temperature and without exerting any pressure. If we extrapolate the graph of pressure P versus temperature T_C (in ºC), the temperature at which the pressure of a gas would be zero is -273.15 ºC. It is seen that all the lines for different gases cut the temperature axis at the same point i.e., at - ??273.15 ºC. This point is termed as the **absolute zero of temperature**. It is not possible to attain a temperature lower than this value. Even to achieve absolute zero temperature is not possible in practice. It may be noted that the point of zero pressure or zero volume does not depend on any specific gas.

The two fixed point scale, described in Section 7.3, had a practical shortcoming for calibrating the scale. It was difficult to precisely control the pressure and identify the fixed points, especially for the boiling point as the boiling temperature is very sensitive to changes in pressure. Hence, a one fixed point scale was adopted in 1954 to define a temperature scale. This scale is called the **absolute scale** or thermodynamic scale. It is named as the kelvin scale after Lord Kelvin (1824-1907).

It is possible for all the three phases - solid, liquid and gas/vapour of a material - to coexist in equilibrium. This is known as the *triple point*. To know the triple point one has to see that three phases coexist in equilibrium and no one phase is dominating. This occurs for each substance at a single unique combination of temperature and pressure. Thus if three phases of water - solid ice, liquid water and water vapour- coexist, the pressure and temperature are automatically fixed. This is termed as the triple point of water and is a single fixed point to define a temperature scale.

The absolute scale of temperature, is so termed since it is based on the properties of an ideal gas and does not depend on the property of any particular substance. The zero of this scale is ideally the lowest temperature possible although it has not been achieved in practice. It is termed as Kelvin scale with its zero at -273.15 °C and temperature intervals same as that on the celsius scale. It is written as K (without °). Internationally, triple point of water has been assigned as 273.16 K at pressure equal to 6.11×10^2 Pa or 6.11×10^{-3} atmosphere, as the standard fixed point for calibration of thermometers. Size of one kelvin is thus 1/273.16 of the difference between the absolute zero and triple point of water. It is same as one celcius. On celcius scale, the triple point of water is 0.01 ºC and not zero.

Three identical thermometers, marked in kelvin, celcius and fahrenheit, placed in a fixed temperature bath, each thermometer showing the same rise in the level of mercury for human body temperature, are depicted in Fig. 7.3.

The relation between the three scales of temperature is as given in Eq. (7.3) .

$$\frac{T_C}{100} = \frac{T_F - 32}{180} = \frac{T_K - 273.15}{100} \quad \text{--- (7.3)}$$

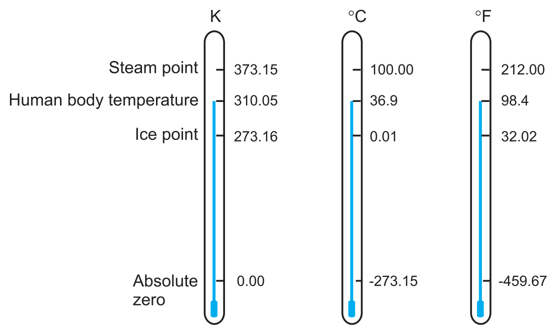


Fig. 7.3: Comparison of the kelvin, celsius and fahrenheit temperature scales (Thermometer reading are not to the scale).

Example 7.5: Express T = 24.57 K in celsius and fahrenheit.

Solution: We have

$$\frac{T_F - 32}{180} = \frac{T_C}{100} = \frac{T_K - 273.15}{100}$$

$$\therefore T_C = T_K - 273.15$$

$$= 24.57 - 273.15$$

$$= -248.58°C$$

$$\frac{T_F - 32}{180} = \frac{T_K - 273.15}{100}$$

$$\therefore T_F = \frac{180}{100}(T_K - 273.15) + 32$$

$$= \frac{9}{5}(24.57 - 273.15) + 32$$

$$= -447.44 + 32$$

$$= -415.44°F$$

Example 7.6: Calculate the temperature which has the same value on fahernheit scale and kelvin scale.

Solution: Let the required temperature be y.

i.e., $T_F = T_K = y$ then we have

$$\frac{y - 32}{180} = \frac{y - 273.15}{100}$$

$$or,\ 5y - 160 = 9y - 2458.35$$

$$or,\ 4y = -160 + 2458.35$$

$$\therefore y = 574.59$$

Thus 574.59 °F and 574.59 K are equivalent temperatures.

7.4.2 Ideal Gas Equation:

The relation between three properties of a gas i.e., pressure, volume and temperature is called ideal gas equation. You will learn more about the properties of gases in chemistry.

Using absolute temperatures, the gas laws can be stated as given below.

1) **Charles' law**- In Fig. 7.2 (b), the volume-temperature graph passes through the origin if temperatures are measured on the kelvin scale, that is if we take 0 K as the origin. In that case the volume V is directly proportional to the absolute temperature T.

   Thus $V \propto T$

   $$\text{or, } \frac{V}{T} = \text{constant} \quad \text{--- (7.4)}$$

   Thus Charles' law can be stated as, the volume of a fixed mass of gas is directly proportional to its absolute temperature if the pressure is kept constant.

2) **Pressure (Gay Lussac's) law**- From Fig.7.2, it can be seen that the pressure-temperature graph is similar to the volume-temperature graph.

   Thus $P \propto T$

   $$\text{or, } \frac{P}{T} = \text{constant} \quad \text{--- (7.5)}$$

   Pressure law can be stated as the pressure of a fixed mass of gas is directly proportional to its absolute temperature if the volume is kept constant.

3) **Boyle's law**- For fixed mass of gas at constant temperature, pressure is inversely proportional to volume.

   $$\text{Thus } P \propto \frac{1}{V}$$

   $$PV = \text{constant} \quad \text{--- (7.6)}$$

   Combining above three equations, we get

   $$\frac{PV}{T} = \text{constant} \quad \text{--- (7.7)}$$

   For one mole of a gas, the constant of proportionality is written as R

   $$\therefore \frac{PV}{T} = R \text{ or } PV = RT \quad \text{--- (7.8)}$$

   If given mass of a gas consists of n moles, then Eq. (7.8) can be written as

   $$PV = nRT \quad \text{--- (7.9)}$$

This relation is called ideal gas equation. The value of constant R is same for all gases. Therefore, it is known as universal gas constant. Its numerical value is 8.31 J K^{-1} mol^{-1}.

Example 7.7: The pressure reading in a thermometer at steam point is 1.367×10^3 Pa. What is pressure reading at triple point knowing the linear relationship between temperature and pressure?

Solution: We have $P_{triple} = 273.16 \times \left(\frac{P}{T}\right)$ where P_{triple} and P are the pressures at temperature of triple point (273.16 K) and T respectively. We are given that $P = 1.367\times10^3$ Pa at steam point i.e., at 273.15 + 100 = 373.15 K.

$$\therefore P_{triple} = 273.16 \times \left(\frac{1.367\times10^3}{373.15}\right)$$

$$= 1.000\times10^3 \text{ Pa}$$

7.5 Thermal Expansion:

When matter is heated, it normally expands and when cooled, it normally contracts. The atoms in a solid vibrate about their mean positions. When heated, they vibrate faster and force each other to move a little farther apart. This results into expansion. The molecules in a liquid or gas move with certain speed. When heated, they move faster and force each other to move a little farther apart. This results in expansion of liquids and gases on heating. The expansion is more in liquids than in solids; gases expand even more.

A change in the temperature of a body causes change in its dimensions. The increase in the dimensions of a body due to an increase in its temperature is called thermal expansion. There are three types of thermal expansion: 1) Linear expansion, 2) Areal expansion, 3) Volume expansion.

7.5.1 Linear Expansion:

The expansion in length due to thermal energy is called linear expansion.

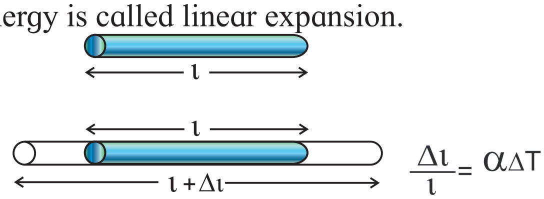


Fig. 7.4: Linear expansion Δl is exaggerated for explanation.

If the substance is in the form of a long rod of length l, then for small change ΔT, in temperature, the fractional change $\Delta l/l$, in length (shown in Fig.7.4), is directly proportional to ΔT.

$$\frac{\Delta l}{l} \propto \Delta T$$

$$\text{or} \quad \frac{\Delta l}{l} = \alpha\, \Delta T \qquad \text{--- (7.10)}$$

where α is called the coefficient of linear expansion of solid. Its value depends upon nature of the material. Rearranging Eq. (7.10), we get

$$\alpha = \frac{\Delta l}{l\Delta T} = \frac{l_T - l_0}{l_0(T - T_0)} \qquad \text{--- (7.11)}$$

where l_0 = length of rod at 0 °C

l_T = length of rod when heated to T °C

T_0 = 0 °C is initial temperature

T = final temperature

$\Delta l = l_T - l_0$ = change in length

$\Delta T = T - T_0$ = rise in temperature

Referring to Eq. (7.11), if l_0=1 and $T - T_0$=1 °C, then

$\alpha = l_T - l_0$ (numerically).

Coefficient of linear expansion of a solid is thus defined as increase in the length per unit original length at 0 °C for one degree centigrade rise in temperature.

The unit of coefficient of linear expansion is per degree celcius or per kelvin. The magnitude of α is very small and it varies only a little with temperature. For most practical purposes, α can be assumed to be constant for a particular material. Therefore, it is not necessary that initial temperature be taken as 0 °C. Equation (7.11) can be rewritten as

$$\alpha = \frac{l_2 - l_1}{l_1(T_2 - T_1)} \qquad \text{--- (7.12)}$$

where l_1 = initial length at temperature T_1 °C

l_2 = final length at temperature T_2 °C.

Table 7.1 lists average values of coefficient of linear expansion for some materials in the temperature range 0°C to 100 °C.

Table 7.1: Values of coefficient of linear expansion for some common materials.

Materials	α (K^{-1})
Carbon (diamond)	0.1×10^{-5}
Glass	0.85×10^{-5}
Iron	1.2×10^{-5}
Steel	1.3×10^{-5}
Gold	1.4×10^{-5}
Copper	1.7×10^{-5}
Silver	1.9×10^{-5}
Aluminium	2.5×10^{-5}
Sulphur	6.1×10^{-5}
Mercury	6.1×10^{-5}
Water	6.9×10^{-5}
Carbon (graphite)	8.8×10^{-5}

Example 7.8: The length of a metal rod at 27 °C is 4 cm. The length increases to 4.02 cm when the metal rod is heated upto 387 °C. Determine the coefficient of linear expansion of the metal rod.

Solution: Given

$T_1 = 27$ °C

$T_2 = 387$ °C

$l_1 = 4$ cm $= 4\times10^{-2}$ m

$l_2 = 4.02$ cm $= 4.02\times10^{-2}$ m

We have

$$\alpha = \frac{l_2 - l_1}{l_1(T_2 - T_1)}$$

$$= \frac{(4.02-4.0)\times10^{-2}}{4\times10^{-2}(387-27)}$$

$$= \frac{0.02\times10^{-2}}{4\times10^{-2}\times360}$$

$$= 1.39\times10^{-5}\ /\text{°C}$$

Example 7.9: Length of an iron rod at temperature 27 °C is 4.256 m. Find the temperature at which the length of the same rod increases to 4.268 m.(α for iron = 1.2×10^{-5} K^{-1})

Solution: Given

$T_1 = 27$ °C, $l_1 = 4.256$ m,

$l_2 = 4.268$ m, $\alpha = 1.2\times10^{-5}$ K^{-1}

We have

$$\alpha = \frac{l_2 - l_1}{l_1(T_2 - T_1)} = \frac{l_2 - l_1}{l_1T_2 - l_1T_1}$$

$$\therefore l_1T_2 = l_1T_1 + \frac{l_2 - l_1}{\alpha}$$

$$T_2 = \frac{1}{l_1}\left[l_1T_1 + \frac{l_2 - l_1}{\alpha}\right]$$

$$= \frac{1}{4.256}\left[(4.256\times27) + \frac{4.268-4.256}{1.2\times10^{-5}}\right]$$

$$= \frac{1}{4.256}\left[114.912 + \frac{0.012}{1.2\times10^{-5}}\right]$$

$$= \frac{1}{4.256}\left[114.912 + 1000\right]$$

$$= 261.96\ \text{°C}$$

7.5.2 Areal Expansion:

The increase ΔA, in the surface area, on heating is called areal expansion or superficial expansion.

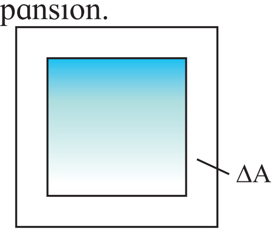


Fig. 7.5: Areal expansion ΔA is exaggerated for explanation.

If a substance is in the form of a plate of area A, then for small change ΔT in temperature, the fractional change in area, $\Delta A/A$ (as shown in Fig. 7.5), is directly proportional to ΔT.

$$\frac{\Delta A}{A} \propto \Delta T$$

or $$\frac{\Delta A}{A} = \beta\,\Delta T \qquad \text{--- (7.13)}$$

where β is called the coefficient of areal expansion of solid. It depends on the material of the solid. Rearranging Eq. (7.13), we get

$$\beta = \frac{\Delta A}{A\Delta T} = \frac{A_T - A_0}{A_0(T - T_0)} \qquad \text{--- (7.14)}$$

where A_0 = area of plate at 0 °C

A_T = area of plate when heated to T °C

T_0 = 0 °C is initial temperature

T = final temperature

$\Delta A = A_T - A_0$ = change in area

$\Delta T = T - T_0$ = rise in temperature.

If A_0 = 1 m^2 and $T - T_0$ = 1 °C, then

$\beta = A_T - A_0$ (numerically).

Therefore, **coefficient of areal expansion of a solid is defined as the increase in the area per unit original area at 0°C for one degree rise in temperature.**

The unit of β is per degree celcius or per kelvin.

As in the case of α, β also does not vary much with temperature. Hence, if A_1 is the area of a metal plate at T_1 °C and A_2 is the area at higher temperature T_2 °C, then

$$\beta = \frac{A_2 - A_1}{A_1(T_2 - T_1)} \quad \text{--- (7.15)}$$

Example 7.10: A thin aluminium plate has an area 286 cm² at 20 °C. Find its area when it is heated to 180 °C.

(β for aluminium = 4.9×10^{-5} /°C)

Solution: Given

$T_1 = 20$ °C
$T_2 = 180$ °C
$A_1 = 286$ cm²
$\beta = 4.9\times10^{-5}$ /°C

We have

$$\beta = \frac{A_2 - A_1}{A_1(T_2 - T_1)}$$

$\therefore A_2 = A_1 [1 + \beta (T_2 - T_1)]$
$= 286 [1 + 4.9\times10^{-5} (180-20)]$
$= 286 [1 + 4.9\times10^{-5}\times160]$
$= 286 [1 + 784.0\times10^{-5}]$
$= 286 [1 + 0.00784]$
$= 286 [1.00784]$
$\therefore A_2 = 288.24$ cm²

7.5.3 Volume expansion

The increase in volume due to heating is called volume expansion or cubical expansion.

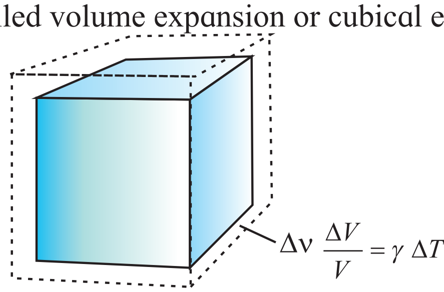


Fig. 7.6: Volume expansion ΔV is exaggerated for explanation.

If the substance is in the form of a cube of volume V, then for small change ΔT in temperature, the fractional change, $\Delta V/V$ (as shown in Fig.7.6), in volume is directly proportional to ΔT.

$$\frac{\Delta V}{V} \propto \Delta T$$

$$\text{or } \frac{\Delta V}{V} = \gamma\,\Delta T \quad \text{--- (7.16)}$$

where γ is called coefficient of cubical or volume expansion. It depends upon the nature of the material. Its unit is per degree celcius or per kelvin. From Eq.(7.16), we can write

$$\gamma = \frac{\Delta V}{V\Delta T} = \frac{V_T - V_0}{V_0(T - T_0)} \quad \text{--- (7.17)}$$

where V_0 = volume at 0 °C
V_T = volume when heated to T °C
$T_0 = 0$ °C is initial temperature
T = final temperature
$\Delta V = V_T - V_0$ = change in volume
$\Delta T = T - T_0$ = rise in temperature.

If $V_0 = 1$ m³, $T - T_0 = 1$ °C, then
$\gamma = V_T - V_0$ (numerically).

The coefficient of cubical expansion of a solid is therefore defined as increase in volume per unit original volume at 0°C for one degree rise in the temperature.

If V_1 is the volume of a body at T_1 °C and V_2 is the volume at higher temperature T_2 °C, then

$$\gamma_1 = \frac{V_2 - V_1}{V_1(T_2 - T_1)} \quad \text{--- (7.18)}$$

γ_1 is the coefficient of volume expansion at temperature T_1 °C.

Since fluids possess definite volume and take the shape of the container, only change in volume is significant. Equations (7.17) and (7.18) are valid for cubical or volume expansion of fluids. It is to be noted that since fluids are kept in containers, when one deals with the volume expansion of fluids, expansion of the container is also to be considered. If expansion of fluid results in a volume greater than the volume of the container, the fluid will overflow if the container is open. If the container is closed, volume expansion of fluid will cause additional pressure on the walls of the container. Can you now tell why the balloon bursts sometimes on its own on a hot day?

Normally solids and liquids expand on heating. Hence their volume increases on heating. Since the mass is constant, it results in a decrease in the density on heating. You have learnt about the anomalous behaviour of water. Water expands on cooling from 4°C to 0°C. Hence its density decreases on cooling in this temperature range.

In Table 7.2 are given typical average values of the coefficient of volume expansion γ for some materials in the temperature range 0°C to 100°C.

Table 7.2: Values of coefficient of volume expansion for some common materials.

Materials	γ (K^{-1})
Invar	2×10^{-6}
Glass (ordinary)	2.5×10^{-5}
Steel	$(3.3\text{-}3.9)\times10^{-5}$
Iron	3.55×10^{-5}
Gold	4.2×10^{-5}
Brass	5.7×10^{-5}
Aluminium	6.9×10^{-5}
Mercury	18.2×10^{-5}
Water	20.7×10^{-5}
Paraffin	58.8×10^{-5}
Gasoline	95.0×10^{-5}
Alcohol (ethyl)	110×10^{-5}

γ is also characteristic of the substance but is not strictly a constant. It depends in general on temperature as shown in Fig.7.7. It is seen that γ becomes constant only at very high temperatures.

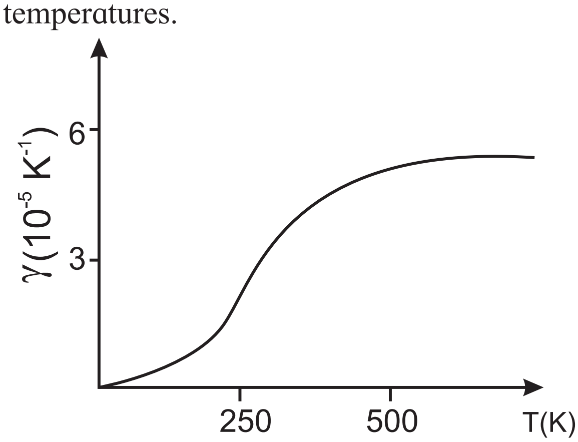


Fig. 7.7: Coefficient of volume expansion of copper as a function of temperature.

Example 7.11 : A liquid at 0 °C is poured in a glass beaker of volume 600 cm³ to fill it completely. The beaker is then heated to 90 °C. How much liquid will overflow?

($\gamma_{liquid} = 1.75\times10^{-4}$/°C, $\gamma_{glass} = 2.75\times10^{-5}$/°C)

Solution: Given

$V_1 = 600$ cm³

$T_1 = 0$ °C

$T_2 = 90$ °C

We have $\gamma = \dfrac{V_2 - V_1}{V_1(T_2 - T_1)}$

$\therefore$ increase is volume $= V_2 - V_1 = \gamma V_1 (T_2 - T_1)$

Increase in volume of beaker

$= \gamma_{glass} \times V_1 (T_2 - T_1)$

$= 2.75\times10^{-5}\times600\times(90-0)$

$= 2.75\times10^{-5}\times600\times90$

$= 148500\times10^{-5}$ cm³

$\therefore$ increase in volume of beaker = 1.485 cm³

Increase in volume of liquid

$= \gamma_{liquid} \times V_1 (T_2 - T_1)$

$= 1.75\times10^{-4}\times600\times(90-0)$

$= 1.75\times10^{-4}\times600\times90$

$= 94500\times10^{-4}$ cm³

$\therefore$ increase in volume of liquid = 9.45 cm³

$\therefore$ volume of liquid which overflows

= (9.45-1.485) cm³

= 7.965 cm³

7.5.4 Relation between Coefficients of Expansion:

i) Relation between β and α:

Consider a square plate of side l_0 at 0 °C and l_T at T °C.

$\therefore l_T = l_0 (1+\alpha T)$ from Eq. (7.11).

If area of plate at 0 °C is A_0, $A_0 = l_0^2$.

If area of plate at T °C is A_T,

$A_T = l_T^2 = l_0^2 (1+\alpha T)^2$

or $A_T = A_0 (1+\alpha T)^2$ --- (7.19)

Also from Eq. (7.14),

$A_T = A_0 (1+\beta T)$ --- (7.20)

Using Eqs. (7.19) and (7.20), we get

$A_0\ (1+\alpha T)^2 = A_0\ (1+\beta T)$

or $1+ 2\alpha T + \alpha^2 T^2 = 1+\beta T$

Since the values of α are very small, the term $\alpha^2 T^2$ is very small and may be neglected.

$\therefore\ \beta = 2\alpha$ --- (7.21)

Can you tell?

1. Why the metal wires for electrical transmission lines sag?
2. Why a railway track is not a continuous piece but is made up of segments separated by gaps?
3. How a steel wheel is mounted on an axle to fit exactly?
4. Why lakes freeze first at the surface?

The result is general because any solid can be regarded as a collection of small squares.

ii) Relation between γ and α:

Consider a cube of side l_0 at 0 °C and l_T at T °C.

$\therefore\ l_T = l_0\ (1+\alpha T)$ from Eq. (7.11).

If volume of the cube at 0 °C is V_0, $V_0 = l_0^{\ 3}$.

If volume of the cube at T °C is V_T,

$V_T = l_T^{\ 3} = l_0^{\ 3}\ (1+\alpha T)^3$

or $V_T = V_0\ (1+\alpha T)^3$ --- (7.22)

Also from Eq. (7.17),

$V_T = V_0\ (1+\gamma T)$ --- (7.23)

Using Eqs. (7.22) and (7.23), we get

$V_0\ (1+\alpha T)^3 = V_0\ (1+\gamma T)$

or $1+ 3\alpha T + 3\alpha^2 T^2 + \alpha^3 T^3 = 1+\gamma T$

Since the values of α are very small, the terms with higher powers of α may be neglected.

$\therefore\ \gamma = 3\alpha$ --- (7.24)

Again the result is general because any solid can be regarded as a collection of small cubes.

Finally, the relation between α, β and γ is

$$\alpha = \frac{\beta}{2} = \frac{\gamma}{3} \quad \text{--- (7.25)}$$

Example 7.12: A sheet of brass is 50 cm long and 8 cm broad at 0 °C. If the surface area at 100 °C is 401.57cm², find the coefficient of linear expansion of brass.

Solution: Given

$T_1 = 0$ °C

$T_2 = 100$ °C

$A_1 = 50\times 8 = 400$ cm²

$A_2 = 401.57$ cm²

We have

$$\beta = 2\alpha = \frac{A_2 - A_1}{A_1(T_2 - T_1)}$$

$$= \frac{(401.57 - 400)\ \text{cm}^2}{400\ \text{cm}^2 \times (100-0)\ °\text{C}}$$

$$= \frac{1.57}{400\times 100} = 0.3925\times 10^{-4}\ °\text{C}^{-1}$$

$$\therefore \alpha = 0.1962\times 10^{-4}\ °\text{C}^{-1}$$

$$= 1.962\times 10^{-5}\ °\text{C}^{-1}$$

∴ Coefficient of linear expansion of brass is 1.962×10^{-5} /°C.

Do you know ?

* When pressure is held constant, due to change in temperature, the volume of a liquid or solid changes very little in comparison to the volume of a gas.
* The coefficient of volume expansion, γ, is generally an order of magnitude larger for liquids than for solids.
* Metals have high values for the coefficient for linear expansion, α, than non-metals.
* γ changes more with temperature than α and β.
* We know that water expands on freezing from 4 ºC to 0 ºC. Other two substances, that expnad on freezing are metals bismuth (Bi) and antimony (Sb). Thus the density of liquid is more than corresponding solid and hence solid Bi or Sb float on their liquids like ice floats on water.

7.6 Specific Heat Capacity:

7.6.1 Specific Heat Capacity of Solids and Liquids

If 1 kg of water and 1 kg of paraffin are heated in turn for the same time by the same heater, the temperature rise of paraffin is about twice that of water. Since the heater gives equal amounts of heat energy to each liquid, it seems that different substances require different amounts of heat to cause the same temperature rise of 1°C in the same mass of 1 kg.

If ΔQ stands for the amount of heat absorbed or given out by a substance of mass m when it undergoes a temperature change ΔT, then the specific heat capacity of that substance is given by

$$s = \frac{\Delta Q}{m\Delta T} \quad \text{--- (7.26)}$$

If m = 1 kg and ΔT = 1°C then $s = \Delta Q$.

Thus specific heat capacity is defined as the amount of heat per unit mass absorbed or given out by the substance to change its temperature by one unit (one degree) 1 °C or 1K.

Table 7.3: Specific heat capacity of some substances at room temperature and atmospheric pressure.

Substance	Specific heat capacity ($J\ kg^{-1}\ K^{-1}$)
Steel	120
Lead	128
Gold	129
Tungsten	134.4
Silver	234
Copper	387
Iron	448
Carbon	506.5
Glass	837
Aluminium	903.0
Kerosene	2118
Paraffin oil	2130
Alcohol (ethyl)	2400
Ethanol	2500
Water	4186.0

The SI unit of specific heat capacity is J/kg °C or J/kg K and C.G.S. unit is erg/g °C or erg/g K. The specific heat capacity is a property of the substance and weakly depends on its temperature. Except for very low temperatures, the specific heat capacity is almost constant for all practical purposes.

If the amount of substance is specified in terms of moles μ instead of mass m in kg, then the specific heat is called molar specific heat (C) and is given by

$$C = \frac{1}{\mu}\frac{\Delta Q}{\Delta T} \quad \text{--- (7.27)}$$

The SI unit of molar specific heat capacity is J/mol °C or J/mol K. Like specific heat, molar specific heat also depends on the nature of the substance and its temperature. Table 7.3 lists the values of specific heat capacity for some common materials.

From Table 7.3, it can be seen that water has the highest specific heat capacity compared to other substances. For this reason, water is used as a coolant in automobile radiators as well as for fomentation using hot water bags.

7.6.2 Specific Heat Capacity of Gas:

In case of a gas, slight change in temperature is accompanied with considerable changes in both, the volume and the pressure. If gas is heated at constant pressure, volume changes and therefore some work is done on the surroundings during expansion requiring additional heat. As a result, specific heat at constant pressure (S_p) is greater than specific heat at constant volume (S_v). It is thus necessary to define two principal specific heat capacities for a gas.

Principal specific heat capacities of gases:

a) The principal specific heat capacity of a gas at constant volume (S_v) is defined as the quantity of heat absorbed or released for the rise or fall of temperature of unit mass of a gas through 1 K (or 1°C) when its volume is kept constant.

b) The principal specific heat capacity of a gas at constant pressure (S_p) is defined as the quantity of heat absorbed or released for the rise or fall of temperature of unit mass of a gas through 1K (1°C) when its pressure is kept constant.

Molar specific heat capacities of gases:

a) Molar specific heat capacity of a gas at constant volume (C_v) is defined as the quantity of heat absorbed or released for the rise or fall of temperature of one mole of the gas through 1K (or 1°C), when its volume is kept constant.

b) Molar specific heat capacity of a gas at constant pressure (C_p) is defined as the quantity of heat absorbed or released for the rise or fall of temperature of one mole of the gas through 1K (or 1°C), when its pressure is kept constant.

Relation between Principal and Molar Specific Heat Capacities:

A relation between principal specific heat capacity and molar specific heat capacity is given by the following expression

Molar specific heat capacity = Molecular weight × principal specific heat capacity.

i.e. $C_p = M \times S_p$ and $C_v = M \times S_v$

where M is the molecular weight of the gas.

Table 7.4 lists values of molar specific heat capacity for some commonly known gases.

Table 7.4: Molar specific heat capacity of some gases.

Gas	C_P (J mol^{-1} K^{-1})	C_V (J mol^{-1} K^{-1})
He	20.8	12.5
H_2	28.8	20.4
N_2	29.1	20.8
O_2	29.4	21.1
CO_2	37.0	28.5

7.6.3 Heat Equation:

If a substance has a specific heat capacity of 1000 J/kg °C, it means that heat energy of 1000 J raises the temperature of 1 kg of that substance by 1°C or 6000 J will raise the temperature of 2 kg of the substance by 3 °C. If the temperature of 2 kg mass of the substance falls by 3 °C, the heat given out would also be 6000 J. In general we can write the heat equation as

Heat received or given out (Q) = mass (m) × temperature change (Δt) × specific heat capacity (s).

or $Q = m \times \Delta T \times s$ --- (7.28)

Example 7.13: If the temperature of 4 kg mass of a material of specific heat capacity 300 J/ kg °C rises from 20 °C to 30 °C. Find the heat received.

Solution:

Q = 4 kg × (30-20) °C × 300 J/kg °C

= 4×10 × 300 J

$\therefore Q$ = 12000 J

7.6.4 Heat Capacity (Thermal Capacity):

Heat capacity or thermal capacity of a body is the quantity of heat needed to raise or lower the temperature of the whole body by 1°C (or 1K).

$\therefore$ Thermal heat capacity can be written as

Heat received or given out

= mass × 1 × specific heat capacity

Heat capacity = $Q = m \times s$ --- (7.29)

Heat capacity (thermal capacity) is measured in J/°C.

Example 7.14: Find thermal capacity for a copper block of mass 0.2 kg, if specific heat capacity of copper is 290 J/kg °C.

Solution: Given

m = 0.2 kg

s = 290 J/kg °C

Thermal capacity = $m \times s$ = 0.2 kg×290 J/kg °C

=58 J/ °C

7.7 Calorimetry:

Calorimetry is an experimental technique for the quantitative measurement of heat exchange. To make such measurement, a calorimeter is used. Figure 7.8 shows a simple water calorimeter.

It consists of cylindrical vessel made of copper or aluminium and provided with a stirrer and a lid. The calorimeter is well-insulated to prohibit any transfer of heat into or out of the calorimeter.

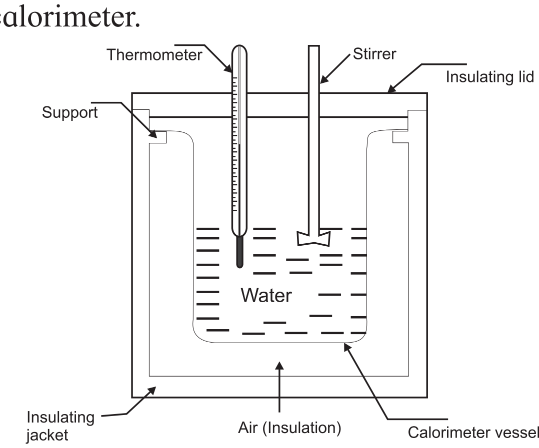


Fig. 7.8: Calorimeter.

One important use of calorimeter is to determine the specific heat of a substance using the principle of conservation of energy. Here we are dealing with heat energy and the system is isolated from surroundings. Therefore, heat gained is equal to the heat lost.

In the technique known as the "method of mixtures", a sample 'A' of the substance is heated to a high temperature which is accurately measured. The sample 'A' is then placed quickly in the calorimeter containing water. The contents are stirred constantly until the mixture attains a final common temperature. The heat lost by the sample 'A' will be gained by the water and the calorimeter. The specific heat of the sample 'A' of the substance can be calculated as under:

Let

m_1 = mass of the sample 'A'

m_2 = mass of the calorimeter and the stirrer

m_3 = mass of the water in calorimeter

s_1 = specific heat capacity of the substance of sample 'A'

s_2 = specific heat capacity of the material of calorimeter (and stirrer)

s_3 = specific heat capacity of water

T_1 = initial temperature of the sample 'A'

T_2 = initial temperature of the calorimeter stirrer and water

T = final temperature of the combined system

We have the data as follows:

Heat lost by the sample 'A' = $m_1 s_1 (T_1 - T)$

Heat gained by the calorimeter and the stirrer

$= m_2 s_2 (T - T_2)$

Heat gained by the water $= m_3 s_3 (T - T_2)$

Assuming no loss of heat to the surroundings, the heat lost by the sample goes into the calorimeter, stirrer and water. Thus writing heat equation as,

$$m_1 s_1 (T_1 - T) = m_2 s_2 (T - T_2) + m_3 s_3 (T - T_2) \quad \text{---(7.30)}$$

Knowing the specific heat capacity of water ($s_3 = 4186$ J kg^{-1} K^{-1}) and copper ($s_2 = 387$ J kg^{-1} K^{-1}) being the material of the calorimeter and the stirrer, one can calculate specific heat capacity (s_1) of material of sample 'A', from Eq. (7.30) as

$$s_1 = \frac{(m_2 s_2 + m_3 s_3)(T - T_2)}{m_1 (T_1 - T)} \quad \text{--- (7.31)}$$

Also, one can find specific heat capacity of water or any liquid using the following expression, it the specific heat capacity of the material of calorimeter and sample is known

$$s_3 = \frac{m_1 s_1 (T_1 - T)}{m_3 (T - T_2)} - \frac{m_2 s_2}{m_3} \quad \text{--- (7.32)}$$

Note - In the experiment, the heat from the solid sample 'A' is given to the liquid and therefore the sample should be denser than the liquid, so that sample does not float on the liquid.

Example 7.15: A sphere of aluminium of 0.06 kg is placed for sufficient time in a vessel containing boiling water so that the sphere is at 100 °C. It is then immediately transferred to 0.12 kg copper calorimeter containing 0.30 kg of water at 25 °C. The temperature of water rises and attains a steady state at 28 °C. Calculate the specific heat capacity of aluminium. (Specific heat capacity of water, $s_w = 4.18 \times 10^3$J kg^{-1} K^{-1}, specific heat capacity of copper $s_{Cu} = 0.387 \times 10^3$ J kg^{-1} K^{-1})

Solution : Given

Mass of aluminium sphere = m_1 = 0.06 kg

Mass of copper calorimeter = m_2 = 0.12 kg

Mass of water in calorimeter

= m_3 = 0.30 kg

Specific heat capacity of copper

$= s_{Cu} = s_2 = 0.387 \times 10^3$ J kg^{-1} K^{-1}

Specific heat capacity of water

$= s_w = s_3 = 4.18 \times 10^3$ J kg^{-1} K^{-1}

Initial temperature of aluminium sphere

$= T_1$ = 100°C

Initial temperature of calorimeter and water = T_2 = 25°C

Final temperature of the mixture

= T = 28°C

We have

$$s_1 = \frac{(m_2 s_2 + m_3 s_3)(T - T_2)}{m_1(T_1 - T)}$$

$$= \frac{\left[(0.12 \times 387) + (0.30 \times 4180)\right](28 - 25)}{(0.06)(100 - 28)}$$

$$= \frac{(46.44 + 1254) \times 3}{(0.06) \times 72} = \frac{3901.32}{4.32}$$

$$= 903.08 \text{ J kg}^{-1} \text{ K}^{-1}$$

$\therefore$ Specific heat capacity of aluminium is 903.08 J kg^{-1} K^{-1}.

7.8 Change of State:

Matter normally exists in three states: solid, liquid and gas. A transition from one of these states to another is called a change of state. Two common changes of states are solid to liquid and liquid to gas (and vice versa). These changes can occur when exchange of heat takes place between the substance and its surroundings.

Activity

To understand the process of change of state

Take some cubes of ice in a beaker. Note the temperature of ice (0 °C). Start heating it slowly on a constant heat source. Note the temperature after every minute. Continuously stir the mixture of water and ice. Observe the change in temperature. Continue heating even after the whole of ice gets converted into water. Observe the change in temperature as before till vapours start coming out. Plot the graph of temperature (along y-axis) versus time (along x-axis). You will obtain a graph of temperature versus time as shown in Fig. 7.9.

Analysis of observations :

1) From point A to B:

There is no change in temperature from point A to point B, this means the temperature of the ice bath does not change even though heat is being continuously supplied. That is the temperature remains constant until the entire amount of the ice melts. The heat supplied is being utilised in changing the state from solid (ice) to liquid (water).

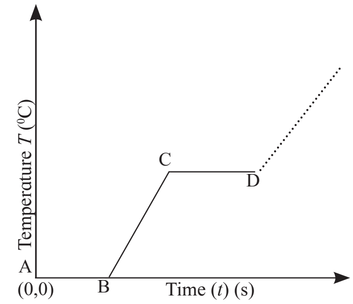


Fig. 7.9 : Variation of temperature with time.

a) The change of state from solid to liquid is called melting and from liquid to solid is called solidification.

b) Both the solid and liquid states of the substance co-exist in thermal equilibrium during the change of states from solid to liquid or vice versa.

c) The temperature at which the solid and the liquid states of the substance are in thermal equilibrium with each other is called the melting point of solid (here ice) or freezing point of liquid (here water). It is characteristic of the substance and also depends on pressure.

d) The melting point of a substance at one standard atmospheric pressure is called its normal melting point.

e) At one standard atmospheric pressure, the freezing point of water and melting point of ice is 0 °C or 32°F. The freezing point describes the liquid to solid transition while melting point describes solid-to-liquid transition.

2) From point B to D:

The temperature begins to rise from point B to point C, i.e., after the whole of ice gets converted into water and we continue further heating. We see that temperature begins to rise. The temperature keeps on rising till it reaches point C i.e., nearly 100 °C. Then it again becomes steady. It is observed that the temperature remains constant until the entire amount of the liquid is converted into vapour. The heat supplied is now being utilized to change water from liquid state to vapour or gaseous state.

a) The change of state from liquid to vapour is called vapourisation while that from vapour to liquid is called condensation.

b) Both the liquid and vapour states of the substance coexist in thermal equilibrium during the change of state from liquid to vapour.

c) The temperature at which the liquid and the vapour states of the substance coexist is called the boiling point of liquid, here water or steam point. This is also the temperature at which water vapour condenses to form water.

d) The boiling point of a substance at one standard atmospheric pressure is called its normal boiling point.

Can you tell?

1. What after point D in graph ? Can steam be hotter than 100 °C ?
2. Why steam at 100 °C causes more harm to our skin than water at 100 °C?

Do you know ?

You must have seen that water spilled on floor dries up after some time. Where does the water disappear? It is converted into water vapour and mixes with air. We say that water has evaporated. You also know that water can be converted into water vapour if you heat the water till its boiling point. **What is then the difference between boiling and evaporation?**

Both evaporation and boiling involve change of state, evaporation can occur at any temperature but boiling takes place at a fixed temperature for a given pressure, unique for each liquid. Evaporation takes place from the surface of liquid while boiling occurs in the whole liquid.

As you know, molecules in a liquid are moving about randomly. The average kinetic energy of the molecules decides the temperature of the liquid. However, all molecules do not move with the same speed. One with higher kinetic energy may escape from the surface region by overcoming the interatomic forces. This process can take place at any temperature. This is evaporation. If the temperature of the liquid is higher, more is the average kinetic energy. Since the number of molecules is fixed, it implies that the number of fast moving molecules is more. Hence the rate of losing such molecules to atmosphere will be larger. Thus, higher is the temperature of the liquid, greater is the rate of evaporation. Since faster molecules are lost, the average kinetic energy of the liquid is reduced and hence the temperature of the liquid is lowered. Hence the phenomenon of evaporation gives a cooling effect to the remaining liquid. Since evaporation takes place from the surface of a liquid, the rate of evaporation is more if the area exposed is more and if the temperature of the liquid is higher.

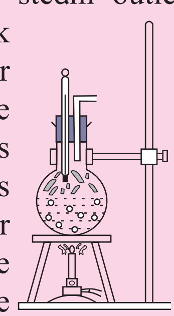
You might have seen that if your mother wants her sari/clothes to dry faster, she does not fold them. More is the area exposed, faster is the drying because the water gets evaporated faster. The presence of wind or strong breeze and content of water vapour in the atmosphere are two other important factors determining the drying of clothes but we do not refer to them here.

Before giving an injection to a patient, normally a spirit swab is used to disinfect the region. We feel a cooling effect on our skin due to evaporation of the spirit as explained before.

Activity

Activity to understand the dependence of boiling point on pressure

Take a round bottom flask, more than half filled with water. Keep it over a burner and fix a thermometer and steam outlet through the cork of the flask as shown in figure. As water in the flask gets heated, note that first the air, which was dissolved in the water comes out as small bubbles. Later bubbles of steam form at the bottom but as they rise to the cooler water near the top, they condense and disappear. Finally, as the temperature of the entire mass of the water reaches 100 °C, bubbles of steam reach the surface and boiling is said to occur. The steam in the flask may not be visible but as it comes out of the flask, it condenses as tiny droplets of water giving a foggy appearance.



If now the steam outlet is closed for a few seconds to increase the pressure in the flask, you will notice that boiling stops. More heat would be required to raise the temperature (depending on the increase in pressure) before boiling starts again. Thus boiling point increases with increase in pressure.

Let us now remove the burner. Allow water to cool to about 80°C. Remove the thermometers and steam outlet. Close the flask with a air tight cork. Keep the flask turned upside down on a stand. Pour ice-cold water on the flask. Water vapours in the flask condense reducing the pressure on the water surface inside the flask. Water begins to boil again, now at a lower temperature. Thus boiling point decreases with decrease in pressure and increases with increase in pressure.

Can you tell?

1. Why cooking is difficult at high altitude?
2. Why cooking is faster in pressure cooker?

7.8.1 Sublimation:

Have you seen what happens when camphor is burnt? All substances do not pass through the three states: solid-liquid-gas. There are certain substances which normally pass from the solid to the vapour state directly and vice versa. The change from solid state to vapour state without passing through the liquid state is called sublimation and the substance is said to sublime. Dry ice (solid CO_2) and iodine sublime. During the sublimation process, both the solid and vapour states of a substance coexist in thermal equilibrium. Most substances sublime at very low pressures.

7.8.2 Phase Diagram:

A pressure - temperature (PT) diagram often called a phase diagram, is particularly convenient for comparing different phases of a substance.

A phase is a homogeneous composition of a material. A substance can exist in different phases in solid state, e.g., you are familiar with two phases of carbon- graphite and diamond. Both are solids but the regular geometric arrangement of carbon atoms is different in the two cases. Figure 7.10 shows the phase diagram of water and CO_2. Let us try to understand the diagram.

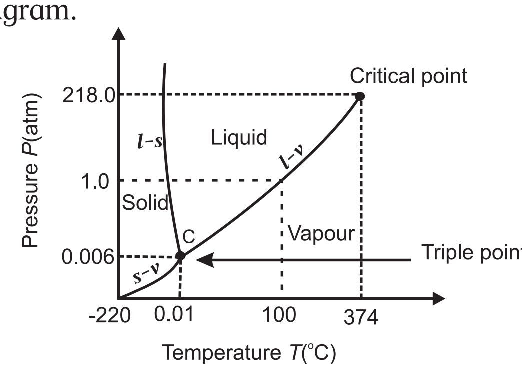


Fig. 7.10 (a): Phase diagram of water (not to scale).

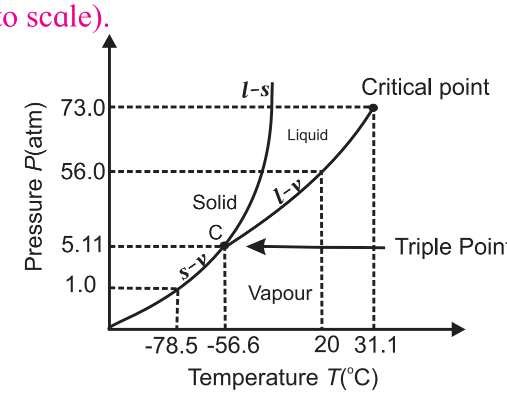


Fig. 7.10 (b): Phase diagram of CO_2 (not to scale).

i) Vapourisation curve *l* - v: The curve labelled *l* - v represents those points where the liquid and vapour phases are in equilibrium. Thus it is a graph of boiling point versus pressure. Note that the curves correctly show that at a pressure of 1 atmosphere, the boiling points of water is 100°C and that the boiling point is lowered for a decreased pressure.

ii) Fusion curve *l* - *s*: The curve *l* - *s* represents the points where the solid and liquid phases coexist in equilibrium. Thus it is a graph of the freezing point versus pressure. At one standard atmosphere pressure, the freezing point of water is 0 °C as shown in Fig. 7.10 (a). Also notice that at a pressure of one standard atmosphere water is in the liquid phase if the temperature is between 0 °C and 100 °C but is in the solid or vapour phase if the temperature is below 0 °C or above 100 °C. Note that *l* - s curve for water slopes upward to the left i.e., fusion curve of water has a slightly negative slope. This is true only of substances that expand upon freezing. However, for most materials like CO_2, the l - s curve slopes upwards to the right i.e., fusion curve has a positive slope. The melting point of CO_2 is -56 °C at higher pressure of 5.11 atm.

iii) Sublimation curve *s* - v: The curve labelled *s* - v is the sublimation point versus pressure curve. Water sublimates at pressure less than 0.0060 atmosphere, while carbon dioxide, which in the solid state is called dry ice, sublimates even at atmospheric pressure at temperature as low as -78 °C.

iv) Triple point: The temperature and pressure at which the fusion curve, the vapourisation curve and the sublimation curve meet and all the three phases of a substance coexist is called the triple point of the substance. That is, the triple point of water is that point where water in solid, liquid and gaseous states coexist in equilibrium and this occurs only at a unique temperature and pressure. The triple point of water is 273.16 K and 6.11×10^{-3} Pa and that of CO_2 is -56.6 °C and 5.1×10^{-5} Pa.

7.8.3 Gas and Vapour:

The terms gas and vapour are sometimes used quite randomly. Therefore, it is important to understand the difference between the two. A gas cannot be liquefied by pressure alone, no matter how high the pressure is. In order to liquefy a gas, it must be cooled to a certain temperature. This temperature is called critical temperature.

Critical temperatures for some common gases and water vapour are given in Table 7.5. Thus, nitrogen must be cooled below -147 °C to liquefy it by pressure.

Table 7.5: Critical Temperatures of some common gases and water vapour.

Gas	Critical Temperature	
	(°C)	(K)
Air	-190	83
N_2	-147	126
O_2	-118	155
CO_2	31.1	241.9
Water vapour	374	647

Gas and vapour can thus be defined as-

1) A substance which is in the gaseous phase and is above its critical temperature is called a gas.

2) A substance which is in the gaseous phase and is below its critical temperature is called a vapour.

Vapour can be liquefied simply by increasing the pressure, while gas cannot. Vapour also exerts pressure like a gas.

7.8.4 Latent Heat:

Whenever there is a change in the state of a substance, heat is either absorbed or given out but there is no change in the temperature of the substance.

Latent heat of a substance is the quantity of heat required to change the state of unit mass of the substance without changing its temperature.

Thus if mass m of a substance undergoes a change from one state to the other then the quantity of heat absorbed or released is given by $Q = mL$ --- (7.33)

where L is known as latent heat and is characteristic of the substance. Its SI unit is J kg^{-1}. The value of L depends on the pressure and is usually quoted at one standard atmospheric pressure.

The quantity of heat required to convert unit mass of a substance from its solid state to the liquid state, at its melting point, without any change in its temperature is called its latent heat of fusion (L_f).

The quantity of heat required to convert unit mass of a substance from its liquid state to vapour state, at its boiling point without any change in its temperature is called its latent heat of vapourization (L_v).

A plot of temperature versus heat energy for a given quantity of water is shown is Fig. 7.11.

From Fig. 7.11, we see that when heat is added (or removed) during a change of state, the temperature remains constant. Also the slopes of the phase lines are not all the same, which indicates that specific heats of the various states are not equal. For water the latent heat of fusion and vaporisation are $L_f = 3.33\times10^5$J kg^{-1} and $L_v = 22.6\times10^5$J kg^{-1} respectively. That is 3.33×10^5J of heat is needed to melt 1kg of ice at 0 °C and 22.6×10^5J of heat is needed to convert 1 kg of water to steam at 100 °C. Hence, steam at 100°C carries 22.6×10^5J kg^{-1} more heat than water at 100 °C. This is why burns from steam are usually more serious than those from boiling water. Melting points, boiling point and latent heats for various substances are given in Table 7.6.

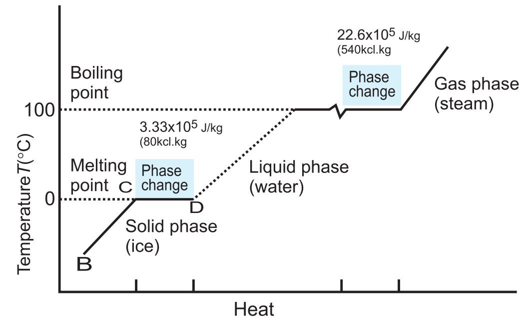


Fig. 7.11: Temperature versus heat for water at one standard atmospheric pressure (not to scale).

Table 7.6 : Temperature of change of state and latent heats for various substances at one standard atmosphere pressure.

Substance	Melting point (°C)	L_f ($\times10^5$ Jkg^{-1})	Boiling point (°C)	L_v ($\times10^5$ Jkg^{-1})
Gold	1063	0.645	2660	15.8
Lead	328	0.25	1744	8.67
Water	0	3.33	100	22.6
Ethyl alcohol	-114	1.0	78	8.5
Mercury	-39	0.12	357	2.7
Nitrogen	-210	0.26	-196	2.0
Oxygen	-219	0.14	-183	2.1

Example 7.16: When 0.1 kg of ice at 0 °C is mixed with 0.32 kg of water at 35 °C in a container. The resulting temperature of the mixture is 7.8 °C. Calculate the heat of fusion of ice (s_{water} = 4186 J kg^{-1} K^{-1}).

Solution: Given

$m_{ice} = 0.1$ kg

$m_{water} = 0.32$ kg

$T_{ice} = 0$ °C

$T_{water} = 35$ °C

$T_F = 7.8$ °C

$s_{water} = 4186$ kg K^{-1}

Heat lost by water

$= m_{water}\, s_{water}\, (T_F - T_{water})$

$= 0.32$ kg $\times$ 4186 J $\times$ (7.8 - 35) °C

$= -\ 36434.944$ J (here negative sign indicates loss of heat energy)

Heat required to melt ice $= m_{ice} L_f = 0.1\times L_f$

Heat required to raise temperature of water (from ice) to final temperature

$= m_{ice}\, s\, (T - T_{ice})$

$= 0.1$ kg$\times$4186 J$\times$(7.8 - 0) C°

$= 3265.08$ J

Head lost = Heat gained

$36434.944 = 0.1\, L_f + 3265.08$

$$L_f = \frac{36434.944 - 3265.08}{0.1} = 3316.9864$$

$$= 3.31698\times10^5 \text{ J kg}^{-1}$$

Do you know ?

The latent heat of vapourization is much larger than the latent heat of fusion. The energy required to completely separate the molecules or atoms is greater than the energy needed to break the rigidity (rigid bonds between the molecules or atoms) in solids. Also when the liquid is converted into vapour, it expands. Work has to be done against the surrounding atmosphere to allow this expansion.

7.9 Heat Transfer:

Heat may be transferred from one point of body to another in three different ways- by conduction, convection and radiation. Heat transferes through solids by conduction. In this process, heat is passed on from one molecule to other molecule but the molecules do not leave their mean positions. Liquids and gases are heated by convection. In this process, there is a bodily movement of the heated molecules. In order to transfer heat by conduction and convection a material medium is required. However transfer of heat by radiation does not need any medium. Radiation of heat energy takes place by electromagnetic (EM) waves that travel with a speed of 3×10^8 ms^{-1} in the space/ vacuum . The energy from the Sun comes to us by radiation. It may be noted that conduction is a slow process of heat transfer while convection is a rapid process. However radiation is the fastest process because the transfer of heat takes place at the speed of light.

7.9.1 Conduction:

Conduction is the process by which heat flows from the hot end to the cold end of a solid body without any net bodily movement of the particles of the body.

Heat passes through solids by conduction only. When one end of a metal rod is placed in a flame while the other end is held in hand, the end held in hand slowly gets hotter, although it itself is not in direct contact with flame. We say that heat has been conducted from the hot end to the cold end. When one end of the rod is heated, the molecules there vibrate faster. As they collide with their slow moving neighbours, they transfer some of their energy by collision to these molecules which in turn transfer energy to their neighbouring molecules still farther down the length of the rod. Thus the energy of thermal motion is transferred by molecular collisions down the rod. The transfer of heat continues till the two ends of the rod are at the same temperature in principle but this will take infinite time. Normally various sections of the rod will attain a temperature which remains constant but not same through out the length of the rod. This method of heat transfer is called conduction.

Those solid substances which conduct heat easily are called good conductors of heat e.g. silver, copper, aluminium, brass etc. All metals are good conductors of heat. Those substances which do not conduct heat easily are called bad conductors of heat e.g. wood, cloth, air, paper, etc. In general, good conductors of heat are also good conductors of electricity. Similarly bad conductors of heat are bad conductors of electricity also.

7.9.1.1 Thermal Conductivity:

Thermal conductivity of a solid is a measure of the ability of the solid to conduct heat through it. Thus good conductors of heat have higher thermal conductivity than bad conductors.

Suppose that one end of a metal rod is heated (see Fig 7.12 (a)). The heat flows by conduction from hot end to the cold end. As a result the temperature of every section of the rod starts increasing. Under this condition, the rod is said to be in a variable temperature state. After some time the temperature at each section of the rod becomes steady i.e. does not change. Note that temperature of each cross-section of the rod is constant but not the same. This is called steady state condition. Under steady state condition, the temperature at points within the rod decreases uniformly with distance from the hot end to the cold end. The fall of temperature with distance between the ends of the rod in the direction of flow of heat, is called temperature gradient.

$$\therefore \text{Temperature gradient} = \frac{T_1 - T_2}{x}$$

where T_1= temperature of hot end

T_2= temperature of cold end

x = length of the rod

7.9.1.2 Coefficient of Thermal Conductivity:

Consider a cube of each side x and each face of cross-sectional area A. Suppose its opposite faces are maintained at temperatures T_1 and T_2 ($T_1 > T_2$) as shown in Fig. 7.12 (b). Experiments show that under steady state condition, the quantity of heat 'Q' that flows from the hot face to the cold face is

i) directly proportional to the cross-sectional area A of the face. i.e., $Q \propto A$

ii) directly proportional to the temperature difference between the two faces i.e., $Q \propto (T_1 - T_2)$

iii) directly proportional to time t (in seconds) for which heat flows i.e. $Q \propto t$

iv) inversely proportional to the perpendicular distance x between hot and cold faces i.e., $Q \propto 1/x$

Combining the above four factors, we have the quantity of heat

$$Q \propto \frac{A(T_1 - T_2)t}{x}$$

$$\therefore Q = \frac{kA(T_1 - T_2)t}{x} \quad \text{--- (7.34)}$$

where k is a constant of proportionality and is called coefficient of thermal conductivity. Its value depends upon the nature of the material.

If $A = 1\ \text{m}^2$, T_1-T_2= 1 °C (or 1 K), t = 1 s and x = 1 m, then from Eq. (7.34), $Q = k$.

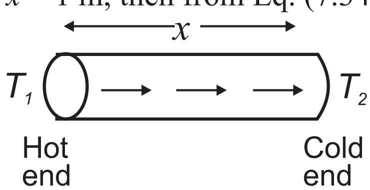


Fig 7.12 (a): Section of a metal bar in the steady state.

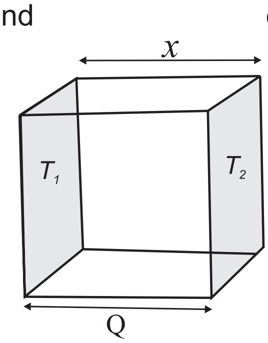


Fig 7.12 (b): Section of a cube in the steady state.

Thus the **coefficient of thermal conductivity of a material is defined as the quantity of heat that flows in one second between the opposite faces of a cube of side 1 m, the faces being kept at a temperature difference of 1°C (or 1 K)**.

From Eq. (7.34), we have

$$k = \frac{Qx}{A(T_1 - T_2)t} \quad \text{--- (7.35)}$$

SI unit of coefficient of thermal conductivity k is J s^{-1} m^{-1} °C^{-1} or J s^{-1} m^{-1} K^{-1} and its dimensions are [L^1 M^1 T^{-3} K^{-1}].

From Eq. (7.34), we also have

$$\frac{Q}{t} = \frac{kA(T_1 - T_2)}{x} \quad \text{--- (7.36)}$$

The quantity Q/t, denoted by P_{cond}, is the time rate of heat flow (i.e. heat flow per second) from the hotter face to the colder face, at right angles to the faces. Its SI unit is watt (W). SI unit of k can therefore be written as W m^{-1} °C^{-1} or W m^{-1} K^{-1}.

Using calculus, Eq. (7.36) may be written as

$$\frac{dQ}{dt} = -kA\frac{dT}{dx}$$

where $\frac{dT}{dx}$ is the temperature gradient.

The negative sign indicates that heat flow is in the direction of decreasing temperature.

If $A = 1\,\text{m}^2$ and $\frac{dT}{dx} = 1$, then $\frac{dQ}{dt} = k$ (numerically).

Hence the **coefficient of thermal conductivity of a material may also be defined as the rate of flow of heat per unit area per unit temperature gradient when the heat flow is at right angles to the faces of a thin parallel-sided slab of material**.

The coefficients of thermal conductivity of some materials are given in Table 7.7.

7.9.1.3 Thermal Resistance (R_T):

Conduction rate P_{cond} is the amount of energy transferred per unit time through a slab of area A and thickness x, the two sides of the slab being at temperatures T_1 and T_2(T_1>T_2), and is given by Eq. (7.36)

$$P_{\text{cond}} = \frac{Q}{t} = kA\frac{T_1 - T_2}{x} \quad \text{--- (7.37)}$$

As discussed earlier, k depends on the material of the slab. A material that readily transfers heat energy by conduction is a good thermal conductor and has high value of k.

Table 7.7: Coefficient of thermal conductivity (k).

Substance	Coefficient of thermal conductivity ($J\ s^{-1}\ m^{-1}\ K^{-1}$)
Silver	406
Copper	385
Aluminium	205
Steel	50.2
Insulating brick	0.15
Glass	0.8
Brick and concrete	0.8
Water	0.8
Wood	0.04-0.12
Air at 0 °C	0.024

In western countries, where the temperature falls below 0 ºC in winter season, insulating the house from the surroundings is very important. In our country, if we wish to carry cold drinks with us for picnic or wish to bring ice-cream from the shop to our house, we need to keep them in containers (made up of say thermocol) that are poor thermal conductors. Hence the concept of thermal resistance R_T, similar to electrical resistance, is introduced. The opposition of a body, to the flow of heat through it, is called thermal resistance. The greater the thermal conductivity of a material, the smaller is its thermal resistance and vice versa. Thus bad thermal conductors are those which have high thermal resistance.

From Eq. (7.37)

$$\frac{(T_1 - T_2)}{P_{cond}} = \frac{x}{kA} \quad \text{--- (7.38)}$$

We know that when a current flows through a conductor, the ratio V/I is called the electrical resistance of the conductor where V is the electrical potential difference between the ends of the conductor and I is the current or rate of flow of charge. In Eq. (7.38), $(T_1 - T_2)$ is the temperature difference between the ends of the conductor and P_{cond} is the rate of flow of heat. Therefore in analogy with electrical resistance, $(T_1 - T_2)/P_{cond}$ is called thermal resistance R_T of the material i.e.,

Thermal resistance $R_T = \dfrac{x}{kA}$

The SI unit of thermal resistance is °C s/ kcal or °C s/J and its dimensional formula is $[M^{-1} L^{-2} T^3 K^1]$.

The lower the thermal conductivity k, the higher is the thermal resistance. R_T A material with high R_T value is a poor thermal conductor and is a good thermal insulator. Thermal resistivity ρ_T is the reciprocal of thermal conductivity k and is characteristic of a material while thermal resistance is that of slab (or of rod) and depends on the material and on the thickness of slab (or length of rod).

Example 7.17: What is the rate of energy loss in watt per square metre through a glass window 5 mm thick if outside temperature is -20 ºC and inside temperature is 25 ºC? (k_{glass} = 1 W/m K)

Solution : Given

k_{glass} = 1 W/m K

T_1 = 25 ºC

T_2 = -20 ºC

x = 5 mm = 5×10^{-3} m

$\therefore\ T_1 - T_2 = 25 - (-20)$ ºC = 45 K

We have $P_{cond} = \dfrac{Q}{t} = kA\dfrac{T_1 - T_2}{x}$.

$\therefore$ The rate of energy loss per square metre is

$$\frac{P_{cond}}{A} = k\frac{T_1 - T_2}{x}$$

$= 1 W\ m^{-1}\ K^{-1} \times 45\ K / (5 \times 10^{-3}\ m)$

$= 9 \times 10^3\ W/m^2$

7.9.1.4 Applications of Thermal Conductivity:

i) Cooking utensils are made of metals but are provided with handles of bad conductors.

Since metals are good conductors of heat, heat can be easily conducted through the base of the utensils. The handles of utensils are made of bad conductors of heat (e.g., wood, ebonite etc.) so that they can not conduct heat from the utensils to our hands.

ii) Thick walls are used in the construction of cold storage rooms. Brick is a bad conductor of heat so that it reduces the flow of heat from the surroundings to the rooms. Still better heat insulation is obtained by using hollow bricks. Air being a poorer conductor than a brick, it further avoids the conduction of heat from outside.

iii) To prevent ice from melting it is wrapped in a gunny bag. A gunny bag is a poor conductor of heat and reduces the heat flow from outside to ice. Moreover, the air filled in the interspaces of a gunny bag, being very bad conductor of heat, further avoids the conduction of heat from outside.

Low thermal conductivity can also be a disadvantage. When hot water is poured in a glass beaker the inner surface of the glass expands on heating. Since glass is a bad conductor of heat, the heat from inside does not reach the outside surface so quickly. Hence the outer surface does not expand thereby causing a crack in the glass.

Example 7.18: The temperature difference between two sides of an iron plate, 2 cm thick, is 10 °C. Heat is transmitted through the plate at the rate of 600 kcal per minute per square metre at steady state. Find the thermal conductivity of iron.

Solution: Given

$$\frac{Q}{At} = 600\ \text{kcal/min m}^2 = \frac{600}{60} = 10\ \text{kcal/s m}^2$$

$$x = 2\ \text{cm} = 2\times10^{-2}\ \text{m}$$

$$T_1 - T_2 = 10\ ^\circ\text{C}$$

From Eq.(7.34), we have

$$Q = \frac{kA(T_1 - T_2)t}{x}$$

$$\therefore k = \frac{Q}{At}\frac{x}{T_1 - T_2}$$

$$= \frac{10\ \text{kcal/s m}^2 \times 2\times10^{-2}\ \text{m}}{10^\circ\text{C}}$$

$$= 0.02\ \text{kcal/m s}\ ^\circ\text{C}$$

Example 7.19: Calculate the rate of loss of heat through a glass window of area 1000 cm^2 and thickness of 4 mm, when temperature inside is 27 °C and outside is -5 °C. Coefficient of thermal conductivity of glass is 0.022 cal/ s cm °C.

Solution : Given

$A = 1000\ \text{cm}^2 = 1000\times10^{-4}\ \text{m}^2$

$k = 0.022\ \text{cal/ s cm}\ ^\circ\text{C} = 0.022\times10^2\ \text{cal/m}\ ^\circ\text{C}$

$x = 4\ \text{mm} = 0.4\times10^{-2}\ \text{m}$

$T_1 = 27\ ^\circ\text{C}$, $T_2 = -5\ ^\circ\text{C}$

From Eq. (7.34), we have

$$Q = \frac{kA(T_1 - T_2)t}{x}$$

$$\therefore \frac{Q}{t} = \frac{kA(T_1 - T_2)}{x}$$

$$= \frac{0.022\times10^2\times1000\times10^{-4}\times(27-(-5))}{0.4\times10^{-2}}$$

$$= 1.76\times10^3\ \text{cal/s} = 1.76\ \text{kcal/s}$$

7.9.2 Convection:

We have seen that heat is transmitted through solids by conduction wherein energy is transferred from one molecule to another but the molecules themselves vibrating with larger amplitude do not leave their mean positions. But in convection, heat is transmitted from one point to another by the actual bodily movement of the heated (energised) molecules within the fluid.

In liquids and gases heat is transmitted by convection because their molecules are quite free to move about. The mechanism of heat transfer by convection in liquids and gases is described below.

Consider water being heated in a vessel from below. The water at the bottom of the vessel is heated first and consequently its density decreases i.e., water molecules at the bottom are separated farther apart. These hot molecules have high kinetic energy and rise upward to cold region while the molecules from cold region come down to take their place. Thus each molecule at the bottom gets heated and rises then cools and descends. This action sets up the flow of water molecules called convection currents. The convection currents transfer heat to the entire mass of water. Note that transfer of heat is by the bodily/ physical movement of the water molecules.

Always remember:

The process by which heat is transmitted through a substance from one point to another due to the actual bodily movement of the heated particles of the substance is called convection.

7.9.2.1 Applications of Convection:

i) Heating and cooling of rooms

The mechanism of heating a room by a heat convector or heater is entirely based on convection. The air molecules in immediate contact with the heater are heated up. These air molecules acquire sufficient energy and rise upward. The cool air at the top being denser moves down to take their place. This cool air in turn gets heated and moves upward. In this way, convection currents are set up in the room which transfer heat to different parts of the room. The same principle but in opposite direction is used to cool a room by an air-conditioner.

ii) Cooling of transformers

Due to current flowing in the windings of the transformer, enormous heat is produced. Therefore, transformer is always kept in a tank containing oil. The oil in contact with transformer body heats up, creating convection currents. The warm oil comes in contact with the cooler tank, gives heat to it and descends to the bottom. It again warms up to rise upward. This process is repeated again and again. The heat of the transformer body is thus carried away by convection to the cooler tank. The cooler tank, in turn loses its heat by convection to the surrounding air.

7.9.2.2 Free and Forced Convection:

i) When a hot body is in contact with air under ordinary conditions, like air around a firewood, the air removes heat from the body by a process called free or natural convection. Land and sea breezes are also formed as a result of free convection currents in air.

ii) The convection process can be accelerated by employing a fan to create a rapid circulation of fresh air. This is called forced convection. Example in section 7.9.2.1 are of forced convection, namely, heat convector, air conditioner, heat radiators in IC engine etc.

7.9.3 Radiation:

The transfer of heat energy from one place to another via emission of EM energy (in a straight line with the speed of light) without heating the intervening medium is called radiation.

For transfer of heat by radiation, molecules are not needed i.e. medium is not required. The fact that Earth receives large quantities of heat form the Sun shows that heat can pass through empty space (i.e., vacuum) between the Sun and the atmosphere that surrounds the Earth . In fact, transfer of heat by radiation has the same properties as light (or EM wave).

A natural question arises as to how heat transfer occurs is the absence of a medium (i.e., molecules). All objects possess thermal energy due to their temperature $T(T > 0 \text{ K})$. The rapidly moving molecules of a hot body emit EM waves travelling with the velocity of light. These are called **thermal radiations**. These carry energy with them and transfer it to the low-speed molecules of a cold body on which they fall. This results in an increase in the molecular motion of the cold body and its temperature rises. Thus transfer of heat by radiation is a two-fold process- the conversion of thermal energy into waves and reconversion of waves into thermal energy by the body on which they fall. We will learn about EM waves in Chapter 13.

7.10 Newton's Laws of Cooling:

If hot water in a vessel is kept on table, it begins to cool gradually. To study how a given body can cool on exchanging heat with its surroundings, following experiment is performed.

A calorimeter is filled up to two third of its capacity with boiling water and is covered. A thermometer is fixed through a hole in the lid and its position is adjusted so that the bulb of the thermometer is fully immersed in water.

The calorimeter vessel is kept in a constant temperature enclosure or just in open air since room temperature will not change much during experiment. The temperature on the thermometer is noted at one minute interval until the temperature of water decreases by about 25 °C. A graph of temperature T (along y-axis) is plotted against time t (along x-axis). This graph is called cooling curve (Fig 7.13 (a)). From this graph you can infer how the cooling of hot water depends on the difference of its temperature from that of its surroundings. You will also notice that initially the rate of cooling is higher and it decreases as the temperature of the water falls. A tangent is drawn to the curve at suitable points on the curve. The slope of each tangent (dT/dt) gives the rate of fall of temperature at that temperature. Taking (0,0) as the origin, if a graph of dT/dt is plotted against corresponding temperature difference (T-T_0), the curve is a straight line as shown in Fig 7.13 (b).

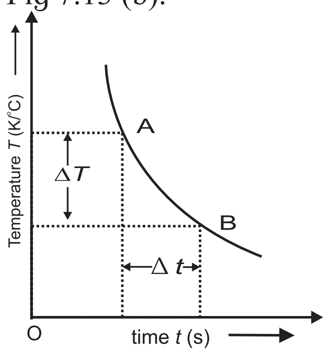


Fig 7.13 (a): Temperature versus time graph. $\lim\limits_{\Delta t \to 0} \frac{\Delta T}{\Delta t}$ gives the slope of the tangent drawn to the curve at point A and indicates the rate of fall of temperature.

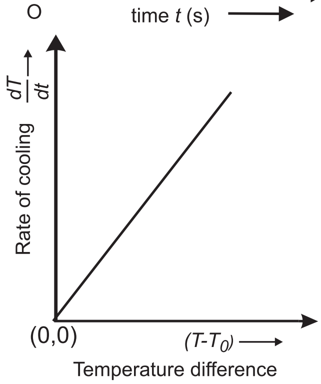


Fig 7.13 (b): Rate of change of temperature versus time graph.

The above activity shows that a hot body loses heat to its surroundings in the form of heat radiation. The rate of loss of heat depends on the difference in the temperature of the body and its surroundings. Newton was the first to study the relation between the heat lost by a body in a given enclosure and its temperature in a systematic manner.

According to Newton's law of cooling the rate of loss of heat dT/dt of the body is directly proportional to the difference of temperature (T -T_0) of the body and the surroundings provided the difference in temperatures is small. Mathematically this may be expressed as

$$\frac{dT}{dt} \propto (T - T_0)$$

$$\therefore \frac{dT}{dt} = C\,(T - T_0) \qquad \text{--- (7.39)}$$

where C is constant of proportionality.

Example 7.20: A metal sphere cools at the rate of 1.6 °C/min when its temperature is 70°C. At what rate will it cool when its temperature is 40°C. The temperature of surroundings is 30°C.

Solution: Given $T_1 = 70°\,C$

$T_2 = 40°\,C$

$T_0 = 30°\,C$

$$\left(\frac{dT}{dt}\right)_1 = 1.6\,°C/\min$$

According to Newton's law of cooling, if C is the constant of proportionality

$$\left(\frac{dT}{dt}\right)_1 = C\,(T_1 - T_0)$$

$$\text{or, } 1.6 = C\,(70 - 30)$$

$$\therefore C = \frac{1.6}{40} = 0.04/\min$$

Also $$\left(\frac{dT}{dt}\right)_2 = C\,(T_2 - T_0)$$

$$= 0.04(40 - 30) = 0.4\,°C/\min$$

Thus the rate of cooling drops by a factor of four when the difference in temperature of the metal sphere and its surroundings drops by a factor of four.

Internet my friend

1. https://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html
2. https://youtu.be/7ZKHc5J6R5Q
3. https://physics.info/expansion

Exercises

1. Choose the correct option.

i) Range of temperature in a clinical thermometer, which measures the temperature of human body, is

(A)70 °C to 100 °C

(B) 34 °C to 42 °C

(C) 0 °F to 100 °F

(D) 34 °F to 80 °F

ii) A glass bottle completely filled with water is kept in the freezer. Why does it crack?

(A) Bottle gets contracted

(B) Bottle is expanded

(C) Water expands on freezing

(D) Water contracts on freezing

iii) If two temperatures differ by 25 °C on Celsius scale, the difference in temperature on Fahrenheit scale is

(A) 65° (B) 45°

(C) 38° (D) 25°

iv) If α, β and γ are coefficients of linear, area *l* and volume expansion of a solid then

(A) $\alpha:\beta:\gamma$ 1:3:2 (B) $\alpha:\beta:\gamma$ 1:2:3

(C) $\alpha:\beta:\gamma$ 2:3:1 (D) $\alpha:\beta:\gamma$ 3:1:2

v) Consider the following statements-

(I) The coefficient of linear expansion has dimension K^{-1}

(II) The coefficient of volume expansion has dimension K^{-1}

(A) I and II are both correct

(B) I is correct but II is wrong

(C) II is correct but I is wrong

(D) I and II are both wrong

vi) Water falls from a height of 200 m. What is the difference in temperature between the water at the top and bottom of a water fall given that specific heat of water is 4200 J kg^{-1} $°C^{-1}$?

(A) 0.96°C (B) 1.02°C

(C) 0.46°C (D) 1.16°C

2. Answer the following questions.

i) Clearly state the difference between heat and temperature?

ii) How a thermometer is calibrated ?

iii) What are different scales of temperature? What is the relation between them?

iv) What is absolute zero?

v) Derive the relation between three coefficients of thermal expansion.

vi) State applications of thermal expansion.

vii) Why do we generally consider two specific heats for a gas?

viii) Are freezing point and melting point same with respect to change of state ? Comment.

ix) Define (i) Sublimation (ii) Triple point.

x) Explain the term 'steady state'.

xi) Define coefficient of thermal conductivity. Derive its expression.

xii) Give any four applications of thermal conductivity in every day life.

xiii) Explain the term thermal resistance. State its SI unit and dimensions.

xiv) How heat transfer occurs through radiation in absence of a medium?

xv) State Newton's law of cooling and explain how it can be experimentally verified.

xvi) What is thermal stress? Give an example of disadvantages of thermal stress in practical use?

xvii) Which materials can be used as thermal insulators and why?

3. Solve the following problems.

i) A glass flask has volume 1×10^{-4} m^3. It is filled with a liquid at 30 °C. If the temperature of the system is raised to 100 °C, how much of the liquid will overflow. (Coefficient of volume expansion of glass is 1.2×10^{-5} $(°C)^{-1}$ while that of the liquid is 75×10^{-5} $(°C)^{-1}$).

[Ans : 516.6×10^{-8} m^3]

ii) Which will require more energy, heating a 2.0 kg block of lead by 30 K or heating a 4.0 kg block of copper by 5 K? (s_{lead} = 128 J kg^{-1} K^{-1}, s_{copper} = 387 J kg^{-1} K^{-1})

[Ans : copper]

iii) Specific latent heat of vaporization of water is 2.26×10^6 J/kg. Calculate the energy needed to change 5.0 g of water into steam at 100 ºC.

[Ans : 11.3×10^3 J]

iv) A metal sphere cools at the rate of 0.05 ºC/s when its temperature is 70 ºC and at the rate of 0.025 ºC/s when its temperature is 50 ºC. Determine the temperature of the surroundings and find the rate of cooling when the temperature of the metal sphere is 40 ºC.

[Ans : 30 ºC, 0.0125 ºC/s]

v) The volume of a gas varied linearly with absolute temperature if its pressure is held constant. Suppose the gas does not liquefy even at very low temperatures, at what temperature the volume of the gas will be ideally zero?

[Ans : -273.15 ºC]

vi) In olden days, while laying the rails for trains, small gaps used to be left between the rail sections to allow for thermal expansion. Suppose the rails are laid at room temperature 27 ºC. If maximum temperature in the region is 45 ºC and the length of each rail section is 10 m, what should be the gap left given that $\alpha = 1.2 \times 10^{-5}$ K^{-1} for the material of the rail section?

[Ans : 2.16 mm]

vii) A blacksmith fixes iron ring on the rim of the wooden wheel of a bullock cart. The diameter of the wooden rim and the iron ring are 1.5 m and 1.47 m respectively at room temperature of 27 ºC. To what temperature the iron ring should be heated so that it can fit the rim of the wheel ($\alpha_{iron} = 1.2\times10^{-5}$ K^{-1}).

[Ans: 1727.7 °C]

viii) In a random temperature scale X, water boils at 200 °X and freezes at 20 °X. Find the boiling point of a liquid in this scale if it boils at 62 °C.

[Ans: 131.6°X]

ix) A gas at 900°C is cooled until both its pressure and volume are halved. Calculate its final temperature.

[Ans: 293.29K]

x) An aluminium rod and iron rod show 1.5 m difference in their lengths when heated at all temperature. What are their lengths at 0 °C if coefficient of linear expansion for aluminium is 24.5×10^{-6} /°C and for iron is 11.9×10^{-6} /°C

[Ans: 1.417m, 2.917m]

xi) What is the specific heat of a metal if 50 cal of heat is needed to raise 6 kg of the metal from 20°C to 62 °C ?

[Ans: s = 0.198 cal/kg °C]

xii) The rate of flow of heat through a copper rod with temperature difference 30 °C is 1500 cal/s. Find the thermal resistance of copper rod.

[Ans: 0.02 °C s cal]

xiii) An electric kettle takes 20 minutes to heat a certain quantity of water from 0°C to its boiling point. It requires 90 minutes to turn all the water at 100°C into steam. Find the latent heat of vaporisation. (Specific heat of water = 1cal/g°C)

[Ans: 450 cal/g]

xiv) Find the temperature difference between two sides of a steel plate 4 cm thick, when heat is transmitted through the plate at the rate of 400 k cal per minute per square metre at steady state. Thermal conductivity of steel is 0.026 kcal/m s K.

[Ans:10.26°C or 10.26 K]

xv) A metal sphere cools from 80 °C to 60 °C in 6 min. How much time with it take to cool from 60 °C to 40 °C if the room temperature is 30°C?

[Ans: 10 min]

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8. Sound

Can you recall?

1. What type of wave is a sound wave?
2. Can sound travel in vacuum?
3. What are reverberation and echo?
4. What is meant by pitch of sound?

8.1 Introduction:

We are all aware of the ripples created on the surface of water when a stone is dropped in it. The water molecules oscillate up and down around their equilibrium positions but they do not move from one point to another along the surface of water. The disturbance created by dropping the stone, however travels outwards. This type of wave is a periodic and regular disturbance in a medium which does not cause any flow of material but causes the flow of energy and momentum from one point to another. There are different types of waves and not all types require material medium to travel through. We know that light is a type of wave and it can travel through vacuum. Here we will first study different types of waves, learn about their common properties and then study sound waves in particular.

Types of waves:

(i) **Mechanical waves:** A wave is said to be mechanical if a material medium is essential for its propagation. Examples of these types of waves are water waves, waves along a stretched string, seismic waves, sound waves, etc.

(ii) **EM waves:** These are generated due to periodic vibrations in electric and magnetic fields. These waves can propagate through material media, however, material medium is not essential for their propagation. These will be studied in Chapter 13.

(iii) **Matter waves:** There is always a wave associated with any object if it is in motion. Such waves are matter waves. These are studied in quantum mechanics.

Travelling or progressive waves are waves in which a disturbance created at one place travels to distant points and keeps travelling unless stopped by some external agencies. In such types of waves energy gets transferred from one point to another. Water waves mentioned above are travelling waves. They keep travelling outward from the point where stone was dropped until they are stopped by walls of the container or the boundary of the water body. Other type of waves are stationary waves about which we will learn in XIIth standard.

8.2 Common Properties of all Waves:

The properties described below are valid for all types of waves, however, here they are described for mechanical waves.

1) **Amplitude** (A): Amplitude of a wave motion is the largest displacement of a particle of the medium through which the wave is propagating, from its rest position. It is measured in metre in SI units.

2) **Wavelength** (λ): Wavelength is the distance between two successive particles which are in the same state of vibration. It is further explained below. It is measured in metre.

3) **Period** (T): Time required to complete one vibration by a particle of the medium is the period T of the wave. It is measured in seconds.

4) **Double periodicity:** Waves possess double periodicity. At every location the wave motion repeats itself at equal intervals of time, hence it is periodic in time. Similarly, at any given instant, the form of wave repeats at equal distances hence, it is periodic in space. In this way wave motion is a doubly periodic phenomenon i.e periodic in time and periodic in space.

5) **Frequency** (n): Frequency of a wave is the number of vibrations performed by a particle during each second. SI unit of frequency is hertz. (Hz) Frequency is a reciprocal of time period, i.e., $n = \frac{1}{T}$

6) Velocity (v): The distance covered by a wave per unit time is called the velocity of the wave. During the period (T), the wave covers a distance equal to the wavelength (λ) Therefore the magnitude of velocity of wave is given by,

$$\text{Magnitude of velocity} = \frac{\text{distance}}{\text{time}}$$

$$\text{v} = \frac{\text{wavelength}}{\text{period}}$$

$$\text{v} = \frac{\lambda}{T} \qquad \text{--- (8.1)}$$

$$\text{but } \frac{1}{T} = n \text{ (frequency)} \qquad \text{--- (8.2)}$$

$$\therefore \text{v} = n\lambda \qquad \text{--- (8.3)}$$

This equation indicates that, the magnitude of velocity of a wave in a medium is constant. Increase in frequency of a wave causes decrease in its wavelength. When a wave goes from one medium to another medium, the frequency of the wave does not change. In such a case speed and wavelength of the wave change.

For mechanical waves to propagate through a medium, the medium should possess certain properties as given below:

i) The medium should be continuous and elastic so that the medium regains original state after removal of deforming forces.

ii) The medium should possess inertia. The medium must be capable of storing energy and of transferring it in the form of waves.

iii) The frictional resistance of the medium must be negligible so that the oscillations will not be damped.

7) **Phase and phase difference:**

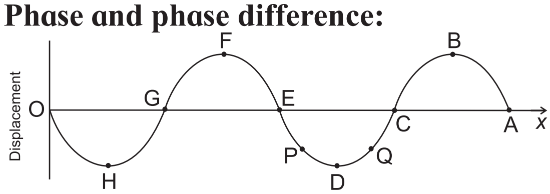


Fig. 8.1 (a): Displacement as a function of distance along the wave.

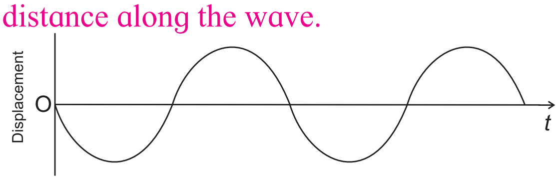


Fig. 8.1 (b): Displacement as a function of time.

In the Fig. 8.1 (a), displacements of various particles along a sinusoidal wave travelling along + ve x-axis are plotted against their respective distances from the source (at O) at a given instant. This plot is valid for transverse as well as longitudinal wave.

The state of oscillation of a particle is called its **phase.** In order to describe the phase at a place, we need to know (a) the displacement (b) the direction of velocity and (c) the oscillation number (during which oscillation) of the particle there.

In Fig. 8.1 (a), particles P and Q (or E and C or B and D) have same displacements but the directions of the their velocities are opposite. Particles B and F have same magnitude of displacements and same direction of velocity. Such particles are said to be in phase during their respective oscillations. Also, these are successive particles with this property of having same phase. Separation between these two particles is wavelength λ. These two successive particles differ by '1' in their oscillation number, i.e., if particle B is at its n^{th} oscillation, particle F will be at its $(n+1)^{th}$ oscillation as the wave is travelling along $+x$ direction. Most convenient way to understand phase is in terms of angle. For a sinusoidal wave, the variation in the displacement is a 'sine' function of distance from the source and of time as discussed below. For such waves it is possible for us to assign angles corresponding to the displacement (or time).

At the instant the above graph is drawn, the disturbance (energy) has just reached the particle A. The phase angle corresponding to this particle A can be taken as 0°. At this instant, particle B has completed quarter oscillation and reached its positive maximum ($\sin\theta = +1$). The phase angle θ of this particle B is $\pi^c/2 = 90°$ at this instant. Similarly, phase angles of particles C and E are π^c (180°) and $2\pi^c$ (360°) respectively. Particle F has completed one oscillation and is at its positive maximum during its second oscillation. Hence its phase angle is $2\pi^c + \frac{\pi^c}{2} = \frac{5\pi^c}{2}$.

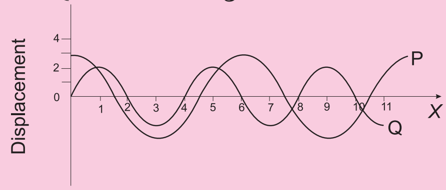
B and F are the successive particles in the same state (same displacement and same direction of velocity) during their respective oscillations. Separation between these two is wavelength (λ). Phase angle between these two differs by $2\pi^c$. Hence wavelength is better understood as the separation between two particles with phase difference of $2\pi^c$.

As noted above, waves possess double periodicity. This means the displacements of particles are periodic in space (as shown in Fig. 8.1 (a)) as well as periodic time. Figure 8.1 (b) shows the displacement of one particular particle as a function of time.

Activity :

(1) Using axes of displacement and distance, sketch two waves A and B such that A has twice the wavelength and half the amplitude of B.

(2) Determine the wavelength and amplitude of each of the two waves P and Q shown in figure below.



Characteristics of progressive wave

1) All vibrating particles of the medium have same amplitude, period and frequency.

2) State of oscillation i.e., phase changes from particle to particle.

Example 8.1: The speed of sound in air is 330 m/s and that in glass is 4500 m/s. What is the ratio of the wavelength of sound of a given frequency in the two media?

Solution: $v_{air} = n\,\lambda_{air}$

$v_{glass} = n\lambda_{glass}$

$$\therefore \frac{\lambda_{air}}{\lambda_{glass}} = \frac{v_{air}}{v_{glass}} = \frac{330}{4500} = 7.33\times10^{-2}$$

$$= 0.0733 \approx 7.33\times10^{-2}$$

8.3 Transverse Waves and Longitudinal Waves:

Progressive waves can be of two types, transverse and longitudinal waves.

Transverse waves : A wave in which particles of the medium vibrate in a direction perpendicular to the direction of propagation of wave is called transverse wave. Water waves are transverse waves, as water molecules vibrate perpendicular to the surface of water while the wave propagates along the surface.

Characteristics of transverse waves.

1) All particles of the medium in the path of the wave vibrate in a direction perpendicular to the direction of propagation of the wave with same period and amplitude.

2) When transverse wave passes through a medium, the medium is divided into alternate the crests i.e., regions of positive displacements and troughs i.e., regions of negative displacements.

3) A crest and an adjacent trough form one cycle of a transverse wave. The distance measured along the wave between any two consecutive points in the same phase (crest or trough) is called the wavelength of the wave.

4) Crests and troughs advance in the medium and are responsible for transfer of energy.

5) Transverse waves can travel through solids and on surfaces of liquids only. They can not travel through liquids and gases. EM waves are transverse waves but they do not require material medium for propagation.

6) When transverse waves advance through a medium there is no change in the pressure and density at any point of medium, however shape changes periodically.

7) If vibrations of all the particles along the path of a wave are constrained to be in a single plane, then the wave is called polarised wave. Transverse wave can be polarised.

8) Medium conveying a transverse wave must possess elasticity of shape.

Longitudinal waves : A wave in which particles of the medium vibrate in a direction parallel to the direction of propagation of wave is called longitudinal wave. Sound waves are longitudinal waves.

Characteristics of longitudinal waves:

1) All the particles of medium along the path of the wave vibrate in a direction parallel to the direction of propagation of wave with same period and amplitude.
2) When longitudinal wave passes through a medium, the medium is divided into regions of alternate compressions and rarefactions. Compression is the region where the particles of medium are crowded (high pressure zone), while rarefaction is the region where the particles of medium are more widely separated, i.e. the medium gets rarefied (low pressure zone).
3) A compression and adjacent rarefaction form one cycle of longitudinal wave. The distance measured along the wave between any two consecutive points having the same phase is the wavelength of wave.
4) For propagation of longitudinal waves, the medium should possess the property of elasticity of volume. Thus longitudinal waves can travel through solids. liquids and gases. Longitudinal wave can not travel through vacuum or free space.
5) The compression and rarefaction advance in the medium and are responsible for transfer of energy.
6) When longitudinal wave advances through a medium there are periodic variations in pressure and density along the path of wave and also with time.
7) Longitudinal waves can not be polarised, as the direction of vibration of particles and direction of propagation of wave are same or parallel.

8.4 Mathematical Expression of a Wave:

Let us describe a progressive wave mathematically. Since it is a progressive wave, we require a function of both the position x and time t. This function will describe the shape of the wave at any instant of time. Another requirement of the function is that it should describe the motion of the particle of the medium at that point. A sinusoidal progressive wave can be described by a sinusoidal function. Let us assume that the progressive wave is transverse and, therefore, the position of the particle of the medium is described by a fixed value of x. The displacement from the equilibrium position can be described by y. Such a sinusoidal wave can be written as follows:

$$y(x,t) = a \sin(kx - \omega t + \phi) \qquad \text{--- (8.4)}$$

Hence a, k, ω and ϕ are constants.

Let us see the justification for writing this equation. At a particular instant say $t = t_o$,

$$y(x, t_0) = a \sin(kx - \omega t_0 + \phi)$$
$$= a \sin(kx + \text{constant})$$

Thus the shape of the wave at $t = t_0$, as a function of x is a sine wave.

Also, at a fixed location $x = x_0$,

$$y(x_0, t) = a \sin(kx_0 - \omega t + \phi)$$
$$= a \sin(\text{constant} - \omega t)$$

Hence the displacement y, at $x = x_0$ varies as a sine function.

This means that the particles of the medium, through which the wave travels, execute simple harmonic motion around their equilibrium position. In addition x must increase in the positive direction as time t increases, so as to keep $(kx - \omega t + \phi)$ a constant. Thus the Eq. (8.4) represents a wave travelling along the positive x axis. A wave represented by

$$y(x, t) = a \sin(kx + \omega t + \phi) \qquad \text{--- (8.5)}$$

is a wave travelling in the direction of the negative x axis.

Symbols in Eq. (8.4):

y (x, t) is the displacement as a function of position (x) and time (t)

a is the amplitude of the wave.

ω is the angular frequency of the wave

k is the angular wave number

$(kx_0 - \omega t + \phi)$ is the argument of the sinusoidal wave and is the phase of the particle at x at time t.

8.5 The Speed of Travelling Waves

Speed of a mechanical wave depends upon the elastic properties and density of the medium. The same medium can support both transverse and longitudinal waves which have different speeds.

8.5.1 The speed of transverse waves

The speed of a wave is determined by the restoring force produced in the medium when it is disturbed. The speed also depends on inertial properties like mass density of the medium. The waves produced on a string are transverse waves. In this case the restoring force is provided by the tension T in the string. The inertial property i.e. the linear mass density m, can be determined from the mass of string M and its length L as $m = M/L$. The formula for speed of transverse wave on stretched string is given by

$$v = \sqrt{\frac{T}{m}} \qquad \text{--- (8.6)}$$

The derivation of the formula is beyond the scope of this book.

The important point here is that the speed of a transverse wave depends only on the properties of the string, T and m. It does not depend on wavelength or frequency of the wave.

8.5.2 The speed of longitudinal waves

In case of longitudinal waves, the particles of the medium oscillate forward and backward along the direction of wave propagation. This causes compression and rarefaction which travel in the medium as the medium possess elastic property.

Speed of sound in liquids and solids is higher than that in gases. The speed of sound as a longitudinal wave in an ideal gas is given by Newton's formula as discussed below.

Speed of sound in different media is given in table below.

Always remember:

When a sound wave goes from one medium to another its velocity changes along with its wavelength. Its frequency, which is decided by the source remains constant.

Table 8.1: Speed of Sound in Gas, Liquids, and Solids

Medium	Speed (m/s)
<u>Gases</u>	
Air [0°C]	331
Air [20°C]	343
Helium	965
Hydrogen	1284
<u>Liquids</u>	
Water (0°C)	1402
Water (20°C)	1482
Seawater	1522
<u>Solids</u>	
Vulcanised Rubber	54
Copper	3560
Steel	5941
Granite	6000
Aluminium	6420

8.5.3 Newton's formula for velocity of sound:

Propagation of longitudinal waves was studied by Newton. Sound waves travel through a medium in the form of compressions and rarefactions. The density of medium is greater at the compression while being smaller in the rarefaction. Hence the velocity of sound depends on elasticity and density of the medium. Newton formulated the relation as

$$v = \sqrt{\frac{E}{\rho}} \qquad \text{--- (8.7)}$$

where E is the proper modulus of elasticity of medium and ρ is the density of medium.

Newton assumed that, during propagation of sound, there is no change in the average temperature of the medium. Hence sound wave propagation in air is an isothermal process (temperature remaining constant) and isothermal elasticity should be considered. The volume elasticity of air determined under isothermal change is called isothermal bulk modulus and is equal to the atmospheric pressure 'P'. Hence Newtons formula for speed of sound in air is given by

$$v = \sqrt{\frac{P}{\rho}} \qquad \text{--- (8.8)}$$

As atmospheric pressure is given by P=hdg and at NTP,

h = 0.76 m of Hg

d = 13600 kg/m^3-density of mercury

ρ = 1.293 kg/m^3- density of air

and g = 9.8 m/s^2

$$v = \sqrt{\frac{0.76 \times 13600 \times 9.8}{1.293}}$$

v = 279.9 m/s at NTP.

This is the value of velocity of sound according to Newton's formula. But the experimental value of velocity of sound at 0^0C as determined earlier by a number of scientists is 332 m/s. The difference between predicted value by Newton's formula and experimental value is large and it is not due to experimental error. The Experimental value is 16% greater than the value given by the formula. Newton could not give satisfactory explanation of this discrepancy. It was resolved by French physicist Pierre Simon Laplace (1749-1827).

Example 8.2: Suppose you are listening to an out-door live concert sitting at a distance of 150 m from the speakers. Your friend is listening to the live broadcast of the concert in another country and the radio signal has to travel 3000 km to reach him. Who will hear the music first and what will be the time difference between the two? Velocity of light =3×10^8 m/s and that of sound is 330m/s.

Solution: Time taken by sound to reach you

$$= \frac{150}{330} s = 0.4546$$

Time taken by the broadcasted sound (which is done by EM waves having velocity =3×10^8m/s)

$$= \frac{3000 \text{ km}}{3\times10^5 \text{ km/s}} = \frac{3\times10^3}{3\times10^5} = 10^{-2} \text{s}$$

$\therefore$ your friend will hear the sound first. The time difference will be

= 0.4546 - 0.01

= 0.4446 s.

8.5.4 Laplace's correction

According to Laplace, the generation of compression and rarefaction is not a slow process but is a rapid process. If frequency is 256 Hz, the air is compressed and rarefied 256 times in a second. Such process must be a rapid process. Heat is produced during compression and is lost during rarefaction. This heat does not get sufficient time for dissipation. Due to this the total heat content remains the same. Such a process is called an adiabatic process and hence, adiabatic elasticity must be adiabatic and not isothermal elasticity, as was assumed by Newton.

Always remember:

In isothermal process temperature remains constant while in adiabatic process there is neither transfer of heat nor of mass.

The adiabatic modulus of elasticity of air is given by,

$$E = \gamma P \qquad \text{--- (8.9)}$$

where P is the pressure of the medium (air) and γ is ratio of specific heat of air at constant pressure (C_p) to the specific heat of air at constant volume (C_v) called as the adiabatic ratio

$$\text{i.e., } \gamma = \frac{C_p}{C_v} \qquad \text{--- (8.10)}$$

For air the ratio of C_p / C_v is 1.41

i.e. $\gamma = 1.41$

Newton's formula for speed of sound in air as modified by Laplace to give

$$v = \sqrt{\frac{\gamma P}{\rho}} \qquad \text{--- (8.11)}$$

Accoriding to this formula velocity of sound at NTP is

$$v = \sqrt{\frac{1.41 \times 0.76 \times 13600 \times 9.8}{1.293}}$$

= 332.3 m/s

This value is in close agreement with the experimental value. As seen above, the velocity of sound depends on the properties of the medium.

8.5.5 Factors affecting speed of sound:

As sound waves travel through atmosphere (open air), some factors related to air affect the speed of sound.

Do you know ?

C_p, the specific heat of gas at constant pressure, is defined as the quantity of heat required to raise the temperature of unit mass of gas through 1^0 K when pressure remains constant.

C_V, the specific heat of gas at constant volume, is defined as the quantity of heat required to raise the temperature of unit mass of gas through 1^0 K when volume remains constant.

When pressure is kept constant the volume of the gas increases with increase in temperature. Thus additional heat is required to increase the volume of gas against the external pressure. Therefore heat required to raise the temperature of unit mass of gas through 1^0 K when pressure is kept constant is greater than the heat required when volume is kept constant. i.e. $C_p > C_V$.

a) Effect of pressure on velocity of sound

According to Laplace's formula velocity of sound in air is

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

If M is the mass and V is volume of air then

$$\rho = \frac{M}{V}$$

$$\therefore v = \sqrt{\frac{\gamma PV}{M}} \qquad \text{--- (8.12)}$$

At constant temperature PV = constant according to Boyle's law. Also M and γ are constant, hence v = constant.

Therefore at constant temperature, a change in pressure has no effect on velocity of sound in air. This can be seen in another way. For gaseous medium, $PV = nRT$, n being the number of moles.

$$\therefore v = \sqrt{\frac{\gamma nRT}{M}} \qquad \text{--- (8.13)}$$

Hence for gaseous medium obeying ideal gas equation change in pressure has no effect on velocity of sound unless there is change in temperature.

Example 8.3: Consider a closed box of rigid walls so that the density of the air inside it is constant. On heating, the pressure of this enclosed air is increased from P_0 to P. It is now observed that sound travels 1.5 times faster than at pressure P_0 calculate P/P_0.

Solution:

$$v_P = \sqrt{\frac{\gamma P_o}{\rho}}$$

$$v_{Po} = \sqrt{\frac{\gamma P_o}{\rho}}$$

$$v_P = 1.5\, v_{Po}$$

$$\sqrt{\frac{\gamma P}{\rho}} = 1.5\sqrt{\frac{\gamma P_o}{\rho}}$$

$$\frac{P}{\rho} = 2.25\frac{P_o}{\rho}$$

$$P = 2.25\, P_o$$

(b) Effect of temperature on speed of sound

Suppose v_o and v are the speeds of sound at T_0 and T in kelvin respectively. Let ρ_0 and ρ be the densities of gas at these two temperatures. The velocity of sound at temperature T_0 and T can be written by using Eq. (8.13),

$$v_0 = \sqrt{\frac{\gamma RT_0}{M}} \quad \text{--- } M \text{ is molar mass, n = 1}$$

$$v = \sqrt{\frac{\gamma RT}{M}}$$

$$\therefore \frac{v}{v_0} = \sqrt{\frac{RT}{RT_0}}$$

$$\therefore \frac{v}{v_0} = \sqrt{\frac{T}{T_0}} \qquad \text{--- (8.14)}$$

This equation shows that speed of sound in air is directly proportional to the square root of absolute temperature. Thus, speed of sound in air increases with increase in temperature. Taking T_o= 273 K and writing T= (273 + t) K where t is the temperature in degree celsius. The ratio of velocity of sound in air at t ^{0}C to that at 0^0C is given by,

$$\therefore \frac{v}{v_0} = \sqrt{\frac{273+t}{273}}$$

$$\therefore \frac{v}{v_0} = \sqrt{1+\frac{t}{273}}$$

$$\therefore \frac{v}{v_0} = \sqrt{1+\alpha t} \quad \text{where } \alpha = \frac{1}{273}$$

or, $v = v_0 (1+\alpha t)^{\frac{1}{2}}$

As α is very small,we can write

$$v \simeq v_0\left(1+\frac{1}{2}\alpha t\right)$$

$$v = v_0\left(1+\frac{1}{2}\times\frac{1}{273}t\right)$$

$$v = v_0\left(1+\frac{t}{546}\right)$$

$$v = v_0 + \frac{v_0}{546}t$$

But v_0 =332 m/s at 0^0C

$$\therefore v = v_0 + \frac{332}{546}t$$

$$\therefore v \simeq v_0 + (0.61)t, \qquad \text{--- (8.15)}$$

i.e., for 1°C rise in temperature velocity increases by 0.61 m/s. Hence for small variations in temperature (< 50° C), the speed of sound changes linearly with temperature.

(c) Effect of humidity on speed of sound

Humidity (moisture) in air depends upon the presence of water vapour in it. Let ρ_m and ρ_d be the densities of moist and dry air respectively. If v_m and v_d are the speeds of sound in moist air and dry air then using Eq. (8.11).

$$v_m = \sqrt{\frac{\gamma P}{\rho_m}}$$

$$\text{and} \quad v_d = \sqrt{\frac{\gamma P}{\rho_d}}$$

$$\therefore \frac{v_m}{v_d} = \sqrt{\frac{\rho_d}{\rho_m}} \qquad \text{--- (8.16)}$$

Moist air is always less dense than dry air, i.e.,

$\rho_m < \rho_d$

(ρ_m = 0.81 kg/m^3(at 0°C) and

ρ_d = 1.29 kg/m^3(at 0°C))

$\therefore v_m > v_d$.

Thus, the speed of sound in moist air is greater than speed of sound in dry air. i.e speed increases with increase in the moistness of air.

8.6 Principle of Superposition of Waves:

Waves don't display any repulsion towards each other. Therefore two wave patterns can overlap in the same region of the space without affecting each other. When two waves overlap their displacements add vectorially. This additive rule is referred to as the principle of superposition of waves.

When two or more waves travelling through a medium arrive at a point of medium simultaneously, each wave produces its own displacement at that point independent of the others. Hence the resultant displacement at that point is equal to the vector sum of the displacements due to all the waves. The phenomenon of superposition will be discussed in detail in XIIth standard.

8.7 Echo, reverberation and acoustics:

Sound waves obey the same laws of reflection as those of light.

8.7.1 Echo:

An echo is the repetition of the original sound because of reflection from some rigid surface at a distance from the source of sound. If we shout in a hilly region, we are likely to hear echo.

Why can't we hear an echo at every place? At 22^{0}C, the velocity of sound in air is 344 m/s. Our brain retains sound for 0.1 second. Thus for us to hear a distinct echo, the sound should take more than 0.1s after starting from the source (i.e., from us) to get reflected and come back to us.

distance = speed × time

= 344 × 0.1

= 34.4 m.

To be able to hear a distinct echo, the reflecting surface should be at a minimum distance of half of the above distance i.e 17.2 m. As velocity depends on the temperature of air, this distance will change with temperature.

Example 8.4: A man shouts loudly close to a high wall. He hears an echo. If the man is at 40 m from the wall, how long after the shout will the echo be heard ? (speed of sound in air = 330 m/s)

solution: The distance travelled by the sound wave

= 2 × distance from man to wall.
= 2 × 40
= 80 m.

$$\therefore \text{Time taken to travel the distance} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{80\text{ m}}{330\text{ m/s}}$$

$$= 0.24\text{ s}$$

∴ The man will hear the echo 0.24 s after he shouts.

8.7.2 Reverberation:

If the reflecting surface is nearer than 15 m from the source of sound, the echo joins up with the original sound which then seems to be prolonged. Sound waves get reflected multiple times from the walls and roof of a closed room which are nearer than 15 m. This causes a single sound to be heard not just once but continuously. This is called reverberation. It is this the persistence of sound after the source has switched off, as a result of repeated reflection from walls, ceilings and other surfaces. Reverberation characteristics are important in the design of concert halls, theatres etc.

If the time between successive reflections of a particular sound wave reaching us is small, the reflected sound gets mixed up and produces a continuous sound of increased loudness which can't be heard clearly.

Reverberation can be decreased by making the walls and roofs rough and by using curtains in the hall to avoid reflection of sound. Chairs and wall surfaces are covered with sound absorbing materials. Porous cardboard sheets, perforated acoustic tiles, gypsum boards, thick curtains etc. at the ceilings and at the walls are most convenient to reduce reverberation.

8.7.3 Acoustics:

The branch of physics which deals with the study of production, transmission and reception of sound is called acoustics. This is useful during the construction of theaters and auditorium. While designing an auditorium, proper care for the absorption and reflection of sound should be taken. Otherwise audience will not be able to hear the sound clearly.

For proper acoustics in an auditorium the following conditions must be satisfied.

1) The sound should be heard sufficiently loudly at all the points in the auditorium. The surface behind the speaker should be parabolic with the speaker at its focus; so that the distribution of sound is uniform in the auditorium. Reflection of sound is helpful in maintaining good loudness through the entire auditorium.
2) Echoes and reverberation must be eliminated or reduced. Echoes can be reduced by making the reflecting surfaces more absorptive. Echo will be less if the auditorium is full.
3) Unnecessary focusing of sound should be avoided and there should not be any zone of poor audibility or region of silence. For that purpose curved surface of the wall or ceiling should be avoided.
4) Echelon effect : It is due to the mixing of sound produced in the hall by the echoes of sound produced in front of regular structure like the stairs. To avoid this, stair type construction must be avoided in the hall.
5) The auditorium should be sound-proof when closed, so that stray sound can not enter from outside.
6) For proper acoustics no sound should be produced from the inside fittings, seats, etc. Instead of fans, air conditioners may be used. Soft action door closers should be used.

Acoustics observed in nature

The importance of acoustic principles goes far beyond human hearing. Several animals use sound for navigation.

(a) Bats depends on sound rather than light to locate objects. So they can fly in total darkness of caves. They emit short ultrasonic pulses of frequency 30 kHz to 150 kHz. The resulting echoes give them information about location of the obstacle.

(b) Dolphins use an analogous system for underwater navigation. The frequencies are subsonic about 100 Hz. They can sense an object of about the size of a wavelength i.e., 1.4 m or larger.

Medical applications of acoustics

(a) Shock waves which are high pressure high amplitude waves are used to split kidney stones into smaller pieces without invasive surgery. A shock wave is produced outside the body and is then focused by a reflector or acoustic lens so that as much of its energy as possible converges on the stone. When the resulting stresses in the stone exceeds its tensile strength, it breaks into small pieces which can be removed easily.

(b) Reflection of ultrasonic waves from regions in the interior of body is used for ultrasonic imaging. It is used for prenatal (before the birth) examination, detection of anamolous conditions like tumour etc and the study of heart valve action.

(c) At very high power level, ultrasound is selective destroyer of pathalogical tissues in treatment of arthritis and certain type of cancer.

Other applications of acoustics

(a) SONAR is an acronym for Sound Navigational Ranging. This is a technique for locating objects underwater by transmitting a pulse of ultrasonic sound and detecting the reflected pulse. The time delay between transmission of a pulse and the reception of reflected pulse indicates the depth of the object. This system is useful to measure motion and position of the submerged objects like submarine.

(b) Acoustic principle has important application to environmental problems like noise control. The design of quiet-mass transit vehicle involves the study of generation and propagation of sound in the motor's wheels and supporting structures.

(c) We can study properties of the Earth by measuring the reflected and refracted elastic waves passing through its interior. It is useful for geological studies to detect local anomalies like oil deposits etc.

8.8 Qualities of sound:

Audible sound or human response to sound:

Whenever we talk about *audible* sound, what matters is how *we perceive* it. This is purely a subjective attribute of sound waves.

Major qualities of sound that are of our interest are (i) Pitch, (ii) Timbre or quality and (iii) Loudness.

(i) Pitch:

This aspect refers to sharpness or shrillness of the sound. If the frequency of sound is increased, what we perceive is the increase in the pitch or we feel the sound to be sharper. *Tone* refers to the single frequency of that wave while a note may contain one or more than one tones. We use the words high pitch or high tones if frequency is higher. As sharpness is a subjective term, sentences like "sound of double frequency is doubly sharp" make no sense. Also, a high pitch sound *need not* be louder. Tones of guitar are sharper than that of a base guitar, sound of tabla is sharper than that of a dagga, (in general) female sound is sharper than that of a male sound and so on.

For a sound amplifier (or equaliser) when we raise the *treble* knob (or *treble* Button), high frequencies are boosted and if we raise *bass* knob, low frequencies are boosted.

(ii) Timbre (sound quality)

During telephonic conversation with a friend, (mostly) you are able to know who is speaking at the other end even if you are not told about who is speaking. Quite often we say, "Couldn't you recognise the voice?" The sound quality in this context is called *timbre*. Same song played on a guitar, a violin, a harmonium or a piano feel significantly different and we can easily identify that instrument. Quality of sound of any sound instrument (including our vocal organ) depends upon the mixture of tones and overtones in the sound generated by that instrument. Even our own sound quality during morning (after we get up) and in the evening is different. It is drastically affected if we are suffering from *cold* or *cough*. Concept of overtones will be discussed during XIIth standard.

(iii) Loudness:

Intensity of a wave is a measurable quantity which is proportional to square of the amplitude ($I \propto A^2$) and is measured in the (SI) unit of W/m^2. Human perception of intensity of sound is *loudness*. Obviously, if intensity is more, loudness is more. The human response to intensity is not linear, i.e., a sound of double intensity is louder but not doubly loud. This is also valid for brightness of light. In both cases, the response is approximately logarithmic. Using this property, the loudness (and brightness) can be measured.

Under ideal conditions, for a perfectly healthy human ear, the least audible intensity is $I_0 = 10^{-12}$ W/m^2. Loudness of a sound of intensity I, measured in the unit ***bel*** is given by

$$L_{\text{bel}} = log_{10}\left(\frac{I}{I_0}\right) \quad \text{--- (8.17)}$$

Popular or commonly used unit for loudness is *deci*bel. We know, 1 *deci*metre or 1 dm = 0.1m. Similarly, 1 decibel or 1 db = 0.1 bel. $\therefore$ 1 bel = 10 db. Thus, loudness expressed in db is 10 times the loudness in bel

$$\therefore L_{\text{db}} = 10\, L_{bel} = 10\, log_{10}\left(\frac{I}{I_0}\right)$$

For sound of least audible intensity I_0,

$$L_{db} = 10\, log_{10}\left(\frac{I_0}{I_0}\right) = 10\, log_{10}(1) = 0 \quad \text{--- (8.18)}$$

This corresponds to threshold of hearing

For sound of 10 db,

$$10 = 10\, log_{10}\left(\frac{I}{I_0}\right) \therefore \left(\frac{I}{I_0}\right) = 10^1 \text{or } I = 10\, I_0$$

For sound of 20 db,

$$20 = 10\, log_{10}\left(\frac{I}{I_0}\right) \therefore \left(\frac{I}{I_0}\right) = 10^2 \text{or } I = 100\, I_0$$

and so on.

Hence, loudness of 20 db sound is *felt* double that of 10 db, but its intensity is 10 times that of the 10 db sound. Now, we feel 40 db sound twice as loud as 20 db sound but its intensity is 100 times as that of 20 db sound and 10000 times that of 10 db sound. This is the power of logarithmic or exponential scale.

If we move away from a (practically) point source, the intensity of its sound varies inversely with square of the distance, i.e., $I \propto \frac{1}{r^2}$.

Whenever you are using earphones or jam your mobile at your ear, the distance from the source is too small. Obviously, such a habit for a long time can affect your normal hearing.

Example 8.5: When heard independently, two sound waves produce sensations of 60 db and 55 db respectively. How much will the sensation be if those are sounded together, perfectly in phase?

Solution:

$$L_1 = 60\, db = 10\, log_{10}\frac{I_1}{I_0} \therefore \frac{I_1}{I_0} = 10^6 \text{ or } I_1 = 10^6\, I_0$$

Similarly, $I_2 = 10^{5.5}\, I_0$

As the waves combine perfectly in phase, the vector addition of their amplitudes will be given by $A^2 = (A_1 + A_2)^2 = A_1^2 + A_2^2 + 2A_1A_2$

As intensity is proportional to square of the amplitude.

$$\therefore I = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

$$= 10^5\, I_0\left(10^1 + 10^{0.5} + 2\sqrt{10^{1.5}}\right)$$

$$= 10^5\, I_0\left(10 + 3.1623 + 2\times 10^{0.75}\right)$$

$$= 24.41\times 10^5\, I_0 = 2.441\times 10^6\, I_0$$

$$\therefore L = 10\, log_{10}\left(\frac{I}{I_0}\right) = 10\, log_{10}\left(2.441\times 10^6\right)$$

$$= 10\left[log_{10}(2.441) + log_{10}\left(10^6\right)\right]$$

$$+ 10(0.3876 = 6)$$

$$= 63.876\,\text{db} \approx 64\text{ db}$$

It is interesting to note that there is only a marginal increase in the loudness.

Table 8.2: Approximate Decibel Ratings of Some Audible Sounds

Source or description of noise	Loudness, L_{db}	Effcet
Extremely loud	160	Immediate ear damage
Jet aeroplane, near 25 m	150	Rupture of eardrum
Auto horn, one metre, Aircraft take off, 60 m	110	Strongly painful
Diesel train, 30 m, Average factory	80	
Highway traffic, 8 m	70	Uncomfortable
Conversion at a restaurant	60	
Conversation at home	50	
Quiet urban background sound	40	
Quiet rural area	30	Virtual silence
Whispering of leaves, 5 m	20	
Normal breathing	10	
Threshold of hearing	0	

8.9 Doppler Effect:

Have you ever heard an approaching train and noticed distinct change in the pitch of the sound of its whistle, when it passes away ? Same thing similar happens when a listener moves towards or away from the stationary source of sound. Such a phenomenon was first identified in 1842 by Austrian physicist Christian Doppler (1803-1853) and is known as Doppler effect.

When a source of sound and a listener are in motion relative to each other the frequency of sound heard by listener is not the same as the frequency emitted by the source.

Doppler effect is the apparent change in frequency of sound due to relative motion between the source and listener. Doppler effect is a wave phenomenon. It holds for sound waves and also for EM waves. But here we shall consider it for sound waves only.

The changes is frequency can be studied under 3 different conditions:

1) When listener is stationary but source is moving.
2) When listener is moving but source is stationary.
3) When listener and source both are moving.

Do you know ?

According to the world health organisation a billion young people could be at risk of hearing loss due to unsafe listening practices. Among teenagers and young adults aged 12-35 years (i) about 50% are exposed to unsafe levels of sound from use of personal audio devices and (ii) about 40% are exposed to potentially damaging sound levels at clubs, discotheques and bars.

8.9.1 Source Moving and Listener Stationary:

Consider a source of sound S, moving away from a stationary listener L (called relative recede) with velocity v_s. Speed of sound waves with respect to the medium is v which is always positive. Suppose the listener uses a detector for counting each wave crest that reaches it.

Initially (at $t = 0$), source which is at point S_1 emits a crest when at distance d from the listener see Fig. 8.2 (a). This crest reaches the listener at time $t_1 = d/v$. Let T_0 be the time period at which the waves are emitted. Thus, at $t = T_0$ the source moves the distance $= v_s T_o$ and reaches the point S_2. Distance of S_2 from the listener is $(d+v_s T_o)$. when at S_2, the source emits second crest. This crest reaches the listener at

$$t_2 = T_0 + \left[\frac{d + v_s T_0}{v}\right] \qquad \text{--- (8.19)}$$

Similarly at time pT_o, the source emits its $(p+1)^{th}$ crest (where, p is an integer, p = 1,2,3,...). It reaches the listener at time

$$t_{p+1} = pT_0 + \left[\frac{d + pv_sT_0}{v}\right]$$

Hence the listener's detector counts p crests in the time interval

$$t_{p+1} - t_1 = pT_0 + \left[\frac{d + pv_sT_0}{v}\right] - \frac{d}{v}$$

Hence the period of wave as recorded by the listener is

$$T = \frac{(t_{p+1} - t_1)}{p} \qquad \text{or}$$

$$T = \frac{\left[pT_0 + \frac{d + pv_sT_0}{v} - \frac{d}{v}\right]}{p}$$

$$T = T_0 + \frac{v_sT_0}{v}$$

$$T = T_0\left[1 + \frac{v_s}{v}\right]$$

$$T = T_0\left[\frac{v + v_s}{v}\right]$$

$$\therefore \frac{1}{T} = \frac{1}{T_0}\left[\frac{v}{v + v_s}\right]$$

$$\therefore n = n_0\left[\frac{v}{v + v_s}\right] \qquad \text{--- (8.20)}$$

where n is the frequency recorded by the listener and n_o is the frequency emitted by the source.

If source of sound is moving towards the listener with speed v_s (called relative approach), the second term from Eq. (8.18) onwards, will be negative (or will be subtracted).

Thus, in this case,

$$n = n_0\left[\frac{v}{v - v_s}\right] \qquad \text{--- (8.21)}$$

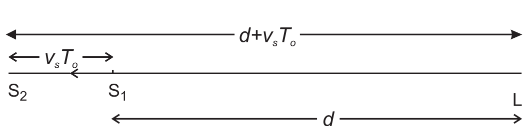


Fig. 8.2 (a): Doppler effect detected when the source is moving and listener is at rest in the medium.

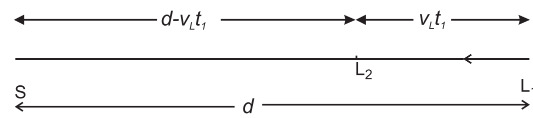


Fig. 8.2 (b): Doppler effect detected when the listener is moving and source is at rest in the medium.

8.9.2 Listener Approaching a Stationary Source with Velocity v_L:

Consider a listener approaching with velocity v_L towards a stationary source S as shown in Fig. 8.2 (b). Let the first wave be emitted by the source at $t = 0$, when the listener was at L_1 at an initial distance d from the source. Let t_1 be the instant when the listener receives this (wave), his position being L_2. During time t_1, the listener travels distance v_Lt_1 towards the stationary source. In this time, the sound wave travels distance $(d - v_Lt_1)$ with speed v.

$$\therefore t_1 = \frac{d - v_Lt_1}{v} \qquad \therefore t_1 = \frac{d}{v + v_L}$$

Second wave is emitted by the source at $t = T_0$ = the time period of the waves emitted by the source. Let t_2 be the instant when the listener receives second wave. During time t_2, the distance travelled by the listener is v_Lt_2. Thus, the distance to be travelled by the sound to reach the listener is then $d - v_Lt_2$.

$\therefore$ Sound (second wave) travels this distance with speed v in time $= \frac{d - v_Lt_2}{v}$

However, this time should be counted after T_0, as the second wave was emitted at $t = T_0$.

$$\therefore t_2 = T_0 + \frac{d - v_Lt_2}{v} \qquad \therefore t_2 = \frac{vT_0 + d}{v + v_L}$$

Similarly, $t_3 = 2T_0 + \frac{d - v_Lt_3}{v} \qquad \therefore t_3 = \frac{2vT_0 + d}{v + v_L}$

Extending this argument to $(p+1)^{th}$ wave, we can write,

$$t_{p+1} = pT_0 + \frac{d - v_Lt_{p+1}}{v} \qquad \therefore t_{p+1} = \frac{pvT_0 + d}{v + v_L}$$

Time duration between instances of receiving successive waves is the *observed* or *recorded* period *T*.

$$\therefore pT = t_{p+1} - t_1 = \frac{pvT_0 + d}{v + v_L} - \frac{d}{v + v_L} = \frac{pvT_0}{v + v_L}$$

$$\therefore T = T_0\left(\frac{v}{v + v_L}\right) \quad \text{--- (8.22)}$$

$$\therefore \frac{1}{T} = \frac{1}{T_0}\left(\frac{v + v_L}{v}\right)$$

$$\therefore n = n_0\left(\frac{v + v_L}{v}\right) \quad \text{--- (8.23)}$$

8.9.3 Both Source and Listener are Moving:

In general when both the source and listener are in motion, we can write the observed frequency

$$n = n_0\left[\frac{v \pm v_L}{v \mp v_s}\right] \quad \text{--- (8.24)}$$

Where the upper signs (in both numerator and denominator) should be chosen during relative approach while lower signs should be chosen during relative recede. It must be remembered that 'when you are deciding the sign for any one of these, the other should be considered to be at rest'.

Illustration:

Consider an observer or listener and a source moving with respective velocities v_L and v_S along the same direction. In this case, listener is approaching the source with v_L (irrespective of whether source is moving or not). Thus, the upper, i.e., positive sign, should be chosen for numerator. However, the source is moving with v_S away form the listener irrespective of listener's motion. Thus the lower sign in the denominator which is positive has to be chosen.

$$\therefore n = n_o\left[\frac{v + v_L}{v + v_s}\right] \quad \text{--- (8.25)}$$

Case (I) If $|v_L| = |v_s|$, $n = n_o$. Thus there is no Doppler shift as there is no relative motion, even if both are moving.

Case (II) If $|v_L| > |v_s|$, numerator will be greater, $n > n_o$. This is because there is relative approach as the listener approaches the source faster and the source is receding at a slower rate.

Case (III) If $|v_L| < |v_s|$, $n < n_o$ as now there is relative recede (source recedes faster, listener approaches slowly).

8.9.4 Common Properties between Doppler Effect of Sound and Light:

A) Wherever there is relative motion between listener (or observer) and source (of sound or light waves), the recorded frequency is different than the emitted frequency.

B) Recorded frequency is higher (than emitted frequency), if there is relative approach.

C) Recorded frequency is lower, if there is relative recede.

D) If v_L or v_s are much smaller then wave speed (speed of sound or light) we can use v_r as relative velocity. In this case, using Eq. (8.24)

$$\frac{\Delta n}{n} \simeq \frac{v_r}{v} \simeq \frac{\Delta\lambda}{\lambda} \quad \text{--- (8.26)}$$

where Δn is Doppler shift or change in the recorded frequency, i.e., $|n - n_o|$ and $\Delta\lambda$ is the recorded change in wavelength.

$$\therefore \frac{|n - n_0|}{n} \simeq \frac{v_r}{v}$$

$$\therefore n = n_0\left(1 \pm \frac{v_r}{v}\right) \quad \text{--- (8.27)}$$

Once again upper sign is to be used during relative approach while lower sign is to be used during relative recede.

E) If velocities of source and observer (listener) are not along the same line their respective components along the line joining them should be chosen for longitudinal Doppler effect and the same mathematical treatment is applicable.

8.9.5 Major Differences between Doppler Effects of Sound and Light:

A) As the speed of light is absolute, only relative velocity between the observer and the source matters, i.e., who is in motion is not relevant.

B) Classical and relativistic Doppler effects are different in the case of light, while in case of sound, it is only classical.

C) For obtaining exact Doppler shift for sound waves, it is absolutely important to know who is in motion.

D) If wind is present, its velocity alters the speed of sound and hence affects the Doppler shift. In this case, component of the wind velocity (v_w) is chosen along the line joining source and observer. This is to be algebraically added with the velocity of sound. Hence 'v' is to be replaced by ($v \pm v_w$) in all the above expressions. Positive sign to be used if v and v_w are along the same direction (remember that v is always positive and always from source to listener). Negative sign is to be used if v and v_w are oppositely directed.

Example 8.6: A rocket is moving at a speed of 220 m/s towards a stationary target. It emits a wave of frequency 1200 Hz. Some of the sound reaching the target gets reflected back to the rocket as an echo. Calculate (1) The frequency of sound detected by the target and (2) The frequency of echo detected by rocket (velocity of sound= 330 m/s.)

Solution: Given, target stationary, i.e.,

$v_L = 0$, $v_s = 220$ m/s, $v = 330$ m/s

$n_0 = 1200$ Hz

To find the frequency of sound detected by the target we have to used Eq. (8.25)

$$n = n_0\left[\frac{v}{v - v_s}\right]$$

$$n = 1200\left[\frac{330}{330 - 220}\right]$$

$$n = 3600 \text{ Hz}$$

The frequency of sound detected by the target = 3600 Hz.

When echo is heard by rocket's detector, target is considered as source

$\therefore v_s = 0$

The frequency of sound emitted by the source (i.e. target) is n_0 = 3600 Hz, and the frequency detected by rocket is n'. Now listener is approaching the source and so we have to use.

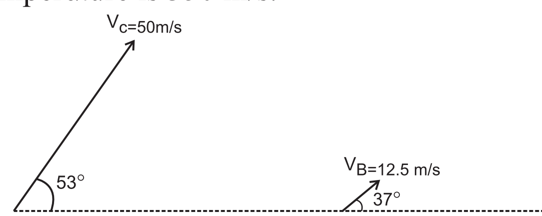
$$n' = n\left[\frac{v + v_L}{v}\right]$$

$$n = 3600\left[\frac{330 + 220}{330}\right]$$

$$n = 6000 \text{ Hz}$$

The frequency of echo detected by rocket = 6000 Hz

Example 8.7: A bat, flying at velocity V_B = 12.5 m/s, is followed by a car running at velocity V_C = 50 m/s. Actual directions of the velocities of the car and the bat are as shown in the figure below, both being in the same horizontal plane (the plane of the figure). To detect the car, the bat radiates ultrasonic waves of frequency 36 kHz. Speed of sound at surrounding temperature is 350 m/s.



There is an ultrasonic frequency detector fitted in the car. Calculate the frequency recorded by this detector.

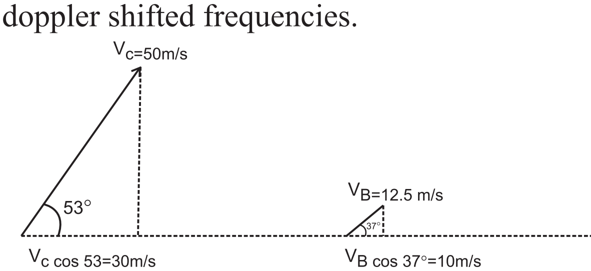
The ultrasonic waves radiated by the bat are reflected by the car. The bat detects these waves and from the detected frequency, it knows about the speed of the car. Calculate the frequency of the reflected waves as detected by the bat. (sin 37° = cos 53° ≈ 0.6, sin 53° = cos 37° ≈ 0.8)

Solution: As shown in the figure below, the components of velocities of the bat and the car, along the line joining them, are

$$V_C \cos 53^0 \cong 50 \times 0.6 = 30 \text{ m s}^{-1} \text{ and}$$

$$V_B \cos 37^0 \cong 12.5 \times 0.8 = 10 \text{ m s}^{-1}.$$

These should be used while calculating the doppler shifted frequencies.



Doppler shifted frequency, $n = n_0\left(\frac{v \pm v_L}{v \mp v_s}\right)$; upper signs to be used during approach, lower signs during recede.

Part I: Frequency radiated by the bat n_0 = 36 $\times 10^3$ Hz, Frequency detected by the detector in the car = n = ?

In this case, bat is the source which is moving away from the car (receding) while the detector in the car is the listener, who is approaching the source (bat). $v_s = V_B \cos 37^0 = 10$ m/s and

$v_L = V_C \cos 53^0 = 30$ m/s

The source (bat) is receding, while the listener (car) is approaching $\therefore n = n_0\left(\frac{v + v_L}{v + v_s}\right)$

$$\therefore n = 36 \times 10^3 \left(\frac{350+30}{350+10}\right)$$

$$= 38 \times 10^3 \text{ Hz} = 38 \text{ kHz}$$

Part II: Reflected frequency, as detected by the bat: Frequency reflected by the car is the Doppler shifted frequency as detected at the car. Thus, this time, the car is the source with emitted frequency $n_0 = 38 \times 10^3$ Hz, $n = ?$

Car, the source, is approaching the listener (bat).

Thus, $v_S = v_C \cos 53^0 = 30$ m/s

Thus, $v_L = v_B \cos 37^0 = 10$ m/s

Now bat-the listener is receding while car the source is approaching $\therefore n = n_0\left(\frac{v - v_L}{v - v_s}\right)$

$$\therefore n = 38 \times 10^3 \left(\frac{350-10}{350-30}\right)$$

$$= 38 \times 10^3 \times \frac{34}{32}$$

$$= 40.375 \times 10^3 \text{ Hz}$$

$$= 40.375 \text{ kHz}$$

Internet my friend

https://hyperphysics.phys-astr.gsu.edu/hbase/hframe.html

Exercises

1. Choose the correct alternatives

i) A sound carried by air from a sitar to a listener is a wave of following type.
(A) Longitudinal stationary
(B)Transverse progressive
(C) Transverse stationary
(D) Longitudinal progressive

ii) When sound waves travel from air to water, which of these remains constant ?
(A) Velocity (B) Frequency
(C) Wavelength (D) All of above

iii) The Laplace's correction in the expression for velocity of sound given by Newton is needed because sound waves
(A) are longitudinal
(B) propagate isothermally
(C) propagate adiabatically
(D) are of long wavelength

iv) Speed of sound is maximum in
(A) air (B) water
(C) vacuum (D) solid

v) The walls of the hall built for music concerns should
(A) amplify sound (B) reflect sound
(C) transmit sound (D) absorb sound

2. Answer briefly.

i) Wave motion is doubly periodic. Explain.
ii) What is Doppler effect?
iii) Describe a transverse wave.
iv) Define a longitudinal wave.
v) State Newton's formula for velocity of sound.
vi) What is the effect of pressure on velocity of sound?
vii) What is the effect of humidity of air on velocity of sound?
viii) What do you mean by an echo?
ix) State any two applications of acoustics.
x) Define amplitude and wavelength of a wave.
xi) Draw a wave and indicate points which are (i) in phase (ii) out of phase (iii) have a phase difference of $\pi/2$.

xii) Define the relation between velocity, wavelength and frequency of wave.

xiii) State and explain principle of superposition of waves.

xiv) State the expression for apparent frequency when source of sound and listener are

i) moving towards each other

ii) moving away from each other

xv) State the expression for apparent frequency when source is stationary and listener is

1) moving towards the source

2) moving away from the source

xvi) State the expression for apparent frequency when listener is stationary and source is.

i) moving towards the listener

ii) moving away from the listener

xvii) Explain what is meant by phase of a wave.

xviii) Define progressive wave. State any four properties.

xix) Distinguish between traverse waves and longitudinal waves.

xx) Explain Newtons formula for velocity of sound. What is its limitation?

3. Solve the following problems.

i) A certain sound wave in air has a speed 340 m/s and wavelength 1.7 m for this wave, calculate

a) the frequency b) the period.

[Ans a) 200 Hz, b) 0.005s]

ii) A tuning fork of frequency 170 Hz produces sound waves of wavelength 2 m. Calculate speed of sound.

[Ans: 340 m/s]

iii) An echo-sounder in a fishing boat receives an echo from a shoal of fish 0.45 s after it was sent. If the speed of sound in water is 1500 m/s, how deep is the shoal?

[Ans : 337.5 m]

iv) A girl stands 170 m away from a high wall and claps her hands at a steady rate so that each clap coincides with the echo of the one before.

a) If she makes 60 claps in 1 minute, what value should be the speed of sound in air?

b) Now, she moves to another location and finds that she should now make 45 claps in 1 minute to coincide with successive echoes. Calculate her distance for the new position from the wall.

[Ans: a) 340 m/s b) 255 m]

v) Sound wave A has period 0.015 s, sound wave B has period 0.025. Which sound has greater frequency?

[Ans : A]

vii) At what temperature will the speed of sound in air be 1.75 times its speed at N.T.P?

[Ans: 836.06 K = 563.06 °C]

viii) A man standing between 2 parallel eliffs fires a gun. He hearns two echos one after 3 seconds and other after 5 seconds. The separation between the two cliffs is 1360 m, what is the speed of sound?

[Ans:340m/s]

ix) If the velocity of sound in air at a given place on two different days of a given week are in the ratio of 1:1.1. Assuming the temperatures on the two days to be same what quantitative conclusion can your draw about the condition on the two days?

[Ans: Air is moist on one day and $\rho_{dry} = 1.1^2 \rho_{dry} = 1.21\ \rho_{moist}$]

x) A police car travels towards a stationary observer at a speed of 15 m/s. The siren on the car emits a sound of frequency 250 Hz. Calculate the recorded frequency. The speed of sound is 340 m/s.

[Ans : 261.54 Hz]

xi) The sound emitted from the siren of an ambulance has frequency of 1500 Hz. The speed of sound is 340 m/s. Calculate the difference in frequencies heard by a stationary observer if the ambulance initially travels towards and then away from the observer at a speed of 30 m/s.

[Ans : 266.79 Hz]

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9. Optics

Can you recall?

1. What are laws of reflection and refraction?
2. What is dispersion of light?
3. What is refractive index?
4. What is total internal reflection?
5. How does light refract at a curved surface?
6. How does a rainbow form?

9.1 Introduction:

"See it to believe it" is a popular saying. In order to see, we need light. What exactly is light and how are we able to see anything? We will explore it in this and next standard. We know that acoustics is the term used for science of sound. Similarly, optics is the term used for science of light. There is a difference in the nature of sound waves and light waves which you have seen in chapter 8 and will learn in chapter 13.

9.2 Nature of light:

Earlier, light was considered to be that form of radiant energy which makes objects visible due to stimulation of retina of the eye. It is a form of energy that propagates in the presence or absence of a medium, which we now call waves. At the beginning of the 20^{th} century, it was proved that these are electromagnetic (EM) waves. Later, using quantum theory, particle nature of light was established. According to this, photons are energy carrier particles. By an experiment using countable number of photons, it is now an established fact that light possesses dual nature. In simple words we can say that light consists of energy carrier photons guided by the rules of EM waves. In vacuum, these waves (or photons) travel with a speed of c = 299792458 m s^{-1} According to Einstein's special theory of relativity, this is the maximum possible speed for any object. For practical purposes we write it as $c = 3\times10^8$ m s^{-1}.

In a material medium, the speed of EM waves is given by $c = \sqrt{\frac{1}{\epsilon\mu}}$,
where permittivity ε and permeability μ are constants which depend on the electric and magnetic properties of the medium.

The ratio $n = \frac{c}{v}$ is called the absolute refractive index and is the property of the medium.

Commonly observed phenomena concerning light can be broadly split into three categories.

(I) **Ray optics or geometrical optics:** A particular direction of propagation of energy from a source of light is called a ray of light. We use ray optics for understanding phenomena like reflection, refraction, double refraction, total internal reflection, etc.

(II) **Wave optics or physical optics:** For explaining phenomena like interference, diffraction, polarization, Doppler effect, etc., we consider light energy to be in the form of EM waves. Wave theory will be further discussed in XII^{th} standard.

(III) **Particle nature of light:** Phenomena like photoelectric effect, emission of spectral lines, Compton effect, etc. cannot be explained by using classical wave theory. These involve the interaction of light with matter. For such phenomena we have to use quantum nature of light. Quantum nature of light will be discussed in XII^{th} standard.

9.3 Ray optics or geometrical optics:

In geometrical optics, we mainly study image formation by mirrors, lenses and prisms. It is based on four fundamental laws/ principles which you have learnt in earlier classes.

(i) Light travels in a straight line in a homogeneous and isotropic medium. Homogeneous means that the properties of the medium are same every where in the medium and isotropic means that the properties are the same in all directions.

(ii) Two or more rays can intersect at a point without affecting their paths beyond that point.

(iii) **Laws of reflection:**

(a) Reflected ray lies in the plane formed by incident ray and the normal drawn at the point of incidence; and the two rays are on either side of the normal.

(b) Angles of incidence and reflection are equal.

(iv) **Laws of refraction:** These apply at the boundary between two media

(a) Refracted ray lies in the plane formed by incident ray and the normal drawn at the point of incidence; and the two rays are on either side of the normal.

(b) Angle of incidence (θ_1 in a medium of refractive index n_1) and angle of refraction (θ_2 in medium of refractive index n_2) are related by Snell's law, given by

$(n_1)\sin\theta_1 = (n_2)\sin\theta_2$.

Do you know ?

Interestingly, all the four laws stated above can be derived from a single principle called Fermat's (pronounced "Ferma") principle. It says that "While travelling from one point to another by one or more reflections or refractions, a ray of light always chooses the path of least time".

Ideally it is the path of extreme time, i.e., path of minimum or maximum time. We strongly recommend you to go through a suitable reference book that will give you the proof of i = r during reflection and Snell's law during refraction using Fermat's principle.

Example 9.1: Thickness of the glass of a spectacle is 2 mm and refractive index of its glass is 1.5. Calculate time taken by light to cross this thickness. Express your answer with the most convenient prefix attached to the unit 'second'.

Solution:

Speed of light in vacuum, $c = 3\times10^8$m/s

$n_{glass} = 10.5$

$\therefore$ Speed of light in glass =

$$\frac{c}{n_{glass}} = \frac{3\times10^8}{1.5} = 2\times10^8 \text{ m / s}$$

Distance to be travelled by light in glass,

s = 2 mm = 2×10^{-3} m

$\therefore$ Time t required by light to travel this distance,

$$t = \frac{s}{\text{v}_{glass}} = \frac{2\times10^{-3}}{2\times10^8} = 10^{-11}\text{s}$$

Most convenient prefix to express this small time is pico (p) = 10^{-12}

$\therefore$ t = 10 × 10^{-12} = 10 ps

9.3.1 Cartesian sign convention:

While using geometrical optics it is necessary to use some sign convention. The relation between only the numerical values of u, v and f for a spherical mirror (or for a lens) will be different for different positions of the object and the type of mirror. Here u and v are the distances of object and image respectively from the optical center, and f is the focal length. Properly used suitable sign convention enables us to use the same formula for all different particular cases. Thus, while deriving a formula and also while using the formula it is necessary to use the same sign convention. Most convenient sign convention is Cartesian sign convention as it is analogous to coordinate geometry. According to this sign convention, (Fig. 9.1):

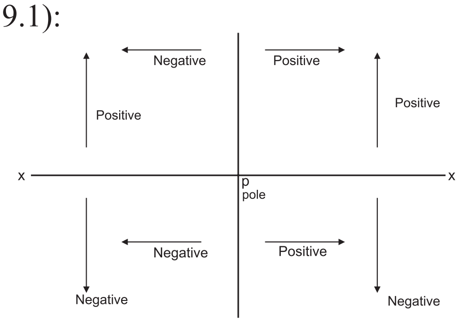


Fig. 9.1 Cartesian sign convention.

i) All distances are measured from the optical center or pole. For most of the optical objects such as spherical mirrors, thin lenses, etc., the optical centers coincides with their geometrical centers.

ii) Figures should be drawn in such a way that the incident rays travel from left to right. A diverging beam of incident rays corresponds to a real point object (Fig. 9.2 (a)), a converging beam of incident rays corresponds to a virtual object (Fig. 9.2 (b)) and a parallel beam corresponds to an object at infinity. Thus, a real object should be shown to the left of pole (Fig. 9.2 (a)) and virtual object or image to the right of pole. (Fig. 9.2 (b))

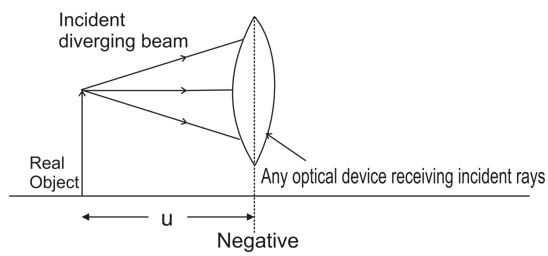


Fig. 9.2: (a) Diverging beam from a real object

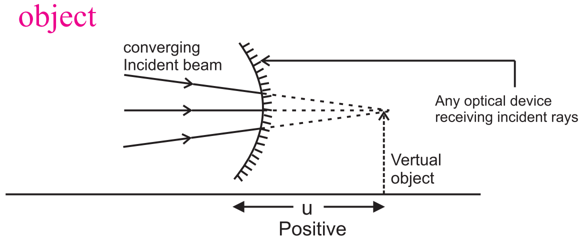


Fig. 9.2: (b) Converging beam towards a virtual object.

iii) x-axis can be conveniently chosen as the principal axis with origin at the pole.

iv) Distances to the left of the pole are negative and those to the right of the pole are positive.

v) Distances above the principal axis (x-axis) are positive while those below it are negative.

Unless specially mentioned, we shall always consider objects to be real for further discussion.

9.4 Reflection:

9.4.1 Reflection from a plane surface:

a) If the object is in front of a plane reflecting surface, the image is virtual and laterally inverted. It is of the same size as that of the object and at the same distance as that of object but on the other side of the reflecting surface.

b) If we are standing on the bank of a still water body and look for our image formed by water (or if we are standing on a plane mirror and look for our image formed by the mirror), the image is laterally reversed, of the same size and on the other side.

c) If an object is kept between two plane mirrors inclined at an angle θ (like in a kaleidoscope), a number of images are formed due to multiple reflections from both the mirrors. Exact number of images depends upon the angle between the mirrors and where exactly the object is kept. It can be obtained as follows (Table 9.1):

Calculate $n = \dfrac{360}{\theta}$

Let N be the number of images seen.

(I) If n is an even integer, $N = (n-1)$, irrespective of where the object is.

(II) If n is an odd integer and object is exactly on the angle bisector, $N = (n-1)$.

(III) If n is an odd integer and object is off the angle bisector, $N = n$

(IV) If n is not an integer, $N = m$, where m is integral part of n.

Table 9.1

Angle θ^0	$n = \dfrac{360}{\theta}$	Position of the object	N
120	3	On angle bisector	2
120	3	Off angle bisector	3
110	3.28	Anywhere	3
90	4	Anywhere	3
80	4.5	Anywhere	4
72	5	On angle bisector	4
72	5	Off angle bisector	5
60	6	Anywhere	5
50	7.2	Anywhere	7

Example 9.2: A small object is kept symmetrically between two plane mirrors inclined at 38°. This angle is now gradually increased to 41°, the object being symmetrical all the time. Determine the number of images visible during the process.

Solution: According to the convention used in the table above,

$$\theta = 38^0 \therefore n = \frac{360}{38} = 9.47$$

$\therefore N = 9$. This is valid till the angle is 40° as the object is kept symmetrically

Beyond 40°, $n < 9$ and it decreases upto

$$\frac{360}{41} = 8.78.$$

Hence now onwards there will be 8 images till 41°.

9.4.2 Reflection from curved mirrors:

In order to focus a parallel or divergent beam by reflection, we need curved mirrors. You might have noticed that reflecting mirrors for a torch or headlights, rear view mirrors of vehicles are not plane but concave or convex. Mirrors for a search light are parabolic. We shall restrict ourselves to spherical mirrors only which can be studied using simple mathematics. Such mirrors are parts of a sphere polished from outside (convex) or from inside (concave).

Radius of the sphere of which a mirror is a part is called as radius of curvature (R) of the mirror. Only for spherical mirrors, half of radius of curvature is focal length of the mirror $\left(f = \frac{R}{2}\right)$. For a concave mirror it is the distance at which parallel incident rays converge. For a convex mirror, it is the distance from where parallel rays appear to be diverging after reflection. According to sign convention, the incident rays are from left to right and they should face the polished surface of the mirror. Thus, focal length of a convex mirror is positive (Fig 9.3 (a)) while that of a concave mirror is negative (Fig. 9.3 (b)).

Relation between *f, u* and *v*:

For a point object or for a small finite object, the focal length of a *small* spherical (concave or convex) mirror is related to object distance and image distance as

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u} \qquad \text{--- (9.1)}$$

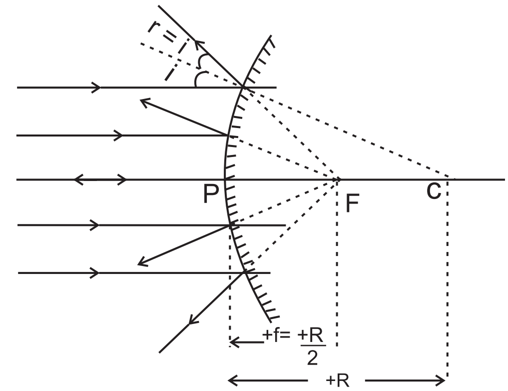


Fig. 9.3 (a): Parallel rays incident from left appear to be diverging from F, lying on the positive side of origin (pole).

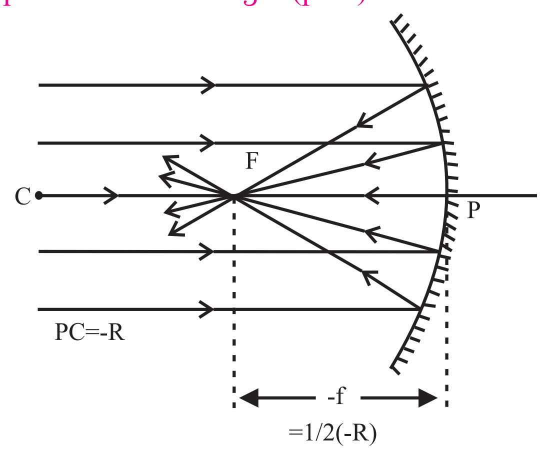


Fig. 9.3 (b): Parallel rays incident from left appear to converge at F, lying on the negative side of origin (pole).

By a *small* mirror we mean its aperture (diameter) is much smaller (at least one tenth) than the values of *u,* v and *f*.

Focal power: Converging or diverging ability of a lens or of a mirror is defined as its focal power. It is measured as $P = \frac{1}{f}$.

In SI units, it is measured as diopter. $\therefore 1\, dioptre(D) = 1m^{-1}$

Lateral magnification: Ratio of linear size of an image to that of the object, measured perpendicular to the principal axis, is defined as the lateral magnification $m = \frac{v}{u}$

For any position of the object, a convex mirror always forms virtual, erect and diminished image, $m<1$. In the case of a concave mirror it depends upon the position of the object. Following Table 9.2 will help you refresh your knowledge.

Table 9.2

Concave mirror (f negative)			
Position of object	**Position of image**	**Real(R) or Virtual (V)**	**Lateral magnifi- -cation**
$u=\infty$	$v=f$	R	$m=0$
$u>2f$	$2f>v>f$	R	$m<1$
$u=2f$	$v=2f$	R	$m=1$
$2f>u>f$	$v>2f$	R	$m>1$
$u=f$	$v=\infty$	R	$m=\infty$
$u<f$	$\lvert v\rvert>\lvert u\rvert$	V	$m>1$

Example 9.3: A thin pencil of length 20 cm is kept along the principal axis of a concave mirror of curvature 30 cm. Nearest end of the pencil is 20 cm from the pole of the mirror. What will be the size of image of the pencil?

Solution: $R = 30$ cm

$f = R/2 = -15$ cm ... (Concave mirror)

$$\frac{1}{f}=\frac{1}{v}+\frac{1}{u}$$

For nearest end, $u = u_1 = -20$ cm . Let the image distance be v_1

$$\therefore \frac{1}{-15}=\frac{1}{v_1}+\frac{1}{-20} \quad \therefore v_1=-60\,\text{cm}$$

Nearest end is at 20 cm and pencil itself is 20 cm long. Hence farthest end is 20 + 20 = 40 cm $=-u_2$

Let the image distance be v_2

$$\therefore \frac{1}{-15}=\frac{1}{v_2}+\frac{1}{-40} \quad \therefore v_2=-24\,\text{cm}$$

$\therefore$ Length of the image = 60 -24 = 36 cm.

Defects or aberration of images: The theory of image formation by mirrors or lenses, and the formulae that we have used such as

$$f=\frac{R}{2} \text{ or } \frac{1}{f}=\frac{1}{v}+\frac{1}{u} \text{ etc.}$$

are based on the following assumptions: (i) Objects and images are situated close to the principal axis.

(ii) Rays diverging from the objects are confined to a cone of very small angle.

(iii) If there is a parallel beam of rays, it is paraxial, i.e., parallel and close to the principal axis.

However, in reality, these assumptions do not always hold good. This results into distorted or defective image. Commonly occurring defects are spherical aberration, coma, astigmatism, curvature, distortion. Except spherical aberration, all the other arise due to beams of rays inclined to principal axis. These are not discussed here.

Spherical aberration: As mentioned earlier, the relation $\left(f=\frac{R}{2}\right)$ giving a single point focus is applicable only for small aperture spherical mirrors and for paraxial rays. In reality, when the rays are farther from the principal axis, the focus gradually shifts towards pole (Fig. 9.4). This phenomenon (defect) arises due to spherical shape of the reflecting surface, hence called as spherical aberration. It results into a unsharp (fuzzy) image with unclear boundaries.

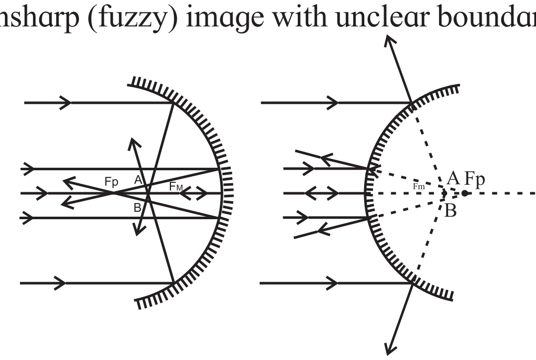


Fig. 9.4: Spherical aberration for curved mirrors.

The distance between F_M and F_P (Fig. 9.4) is measured as the longitudinal spherical aberration. If there is no spherical aberration, we get a single point image on a screen placed perpendicular to the principal axis at that location, for a beam of incident rays parallel to the axis. In the presence of spherical aberration, no such point is possible at any position of the screen and the image is always a circle. At a particular location of the screen, the diameter of this circle is minimum. This is called the circle of least confusion. In the figures it is across AB. Radius of this circle is transverse spherical aberration.

In the case of curved mirrors, this defect can be completely eliminated by using a parabolic mirror. Hence surfaces of mirrors used in a search light, torch, headlight of a car, telescopes, etc., are parabolic and not spherical.

Do you know ?

Why does a parabolic mirror not have spherical aberration?

Parabola is a geometrical shape drawn in such a way that every point on it is equidistant from a straight line and from a point. Figure 9.5 shows a parabola. Points A, B, C, … on it are equidistant from line RS (called directrix) and point F (called focus). Hence $A'A = AF$, $B'B = BF$, $C'C = CF$, ….

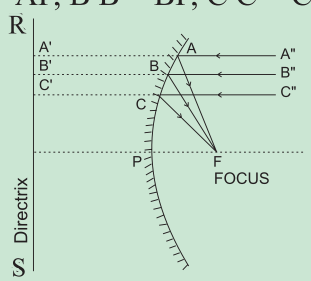


Fig. 9.5: Single focus for parabolic mirror.

If rays of equal optical path converge at a point, that point is the location of real image corresponding to that beam of rays.

Paths A″AA′, B″BB′. C″CC′, etc., are equal paths in the absence of mirror. If the parabola ABC… is a mirror then the respective optical paths will be A″AF, B″BF, C″CF, … and from the definition of parabola, these are also equal. Thus, F is the single point focus for entire beam parallel to the axis with NO spherical aberration.

9.5 Refraction:

Being an EM wave, the properties of light (speed, wavelength, direction of propagation, etc.) depend upon the medium through which it is traveling. If a ray of light comes to an interface between two media and enters into another medium of different refractive index, it changes itself suitable to that medium. This phenomenon is defined as refraction of light. The extent to which these properties change is decided by the index of refraction, 'n'.

Absolute refractive index:

Absolute refractive index of a medium is defined as the ratio of speed of light in vacuum to that in the given medium.

$n = \frac{c}{v}$ where c and v are respective speeds of light in vacuum and in the medium. As n is the ratio of same physical quantities, it is a unitless and dimensionless physical quantity.

For any material medium (including air) $n > 1$, i.e., light travels fastest in vacuum than in any material medium. Medium having greater value of n is called optically denser. An optically denser medium need not be physically denser, e.g., many oils are optically denser than water but water is physically denser than them.

Relative refractive index:

Refractive index of medium 2 with respect to medium 1 is defined as the ratio of speed of light v_1 in medium 1 to its speed v_2 in medium 2. Thus, ${}^1n_2 = \frac{n_2}{n_1} = \frac{v_1}{v_2}$

Do you know ?

(a) Logic behind the convention ${}^1n_2$: Letter n is the symbol for refractive index, n_2 corresponds to refractive index of medium 2 and ${}^1n_2$ indicates that it is with respect to medium 1. In this case, light travels from medium 1 to 2 so we need to discuss medium 2 in context to medium 1.

(b) Dictionary meaning of the word refract is to change the path`. However, in context of Physics, we should be more specific. We use the word *deviate* for changing the path. During refraction at normal incidence, there is no change in path. Thus, there is *refraction* but no *deviation*. Deviation is associated with refraction *only* during oblique incidence. Deviation or changing the path or bending is associated with many phenomena such as reflection, diffraction, scattering, gravitational bending due to a massive object, etc.

Illustrations of refraction: 1) When seen from outside, the bottom of a water body appears to be raised. This is due to refraction at the plane surface of water. In this case,

$$n_{water} \cong \frac{Real\ depth}{apparent\ depth}$$

This relation holds good for a plane parallel transparent slab also as shown below.

Figure 9.6 shows a plane parallel slab of a transparent medium of refractive index *n*. A point object O at real depth *R* appears to be at I at apparent depth *A*, when seen from outside (air). Incident rays OA (traveling undeviated) and OB (deviating along BC) are used to locate the image.

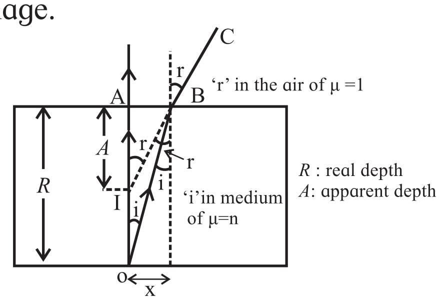


Fig. 9.6: Real and apparent depth.

By considering i and r to be small, we can write,

$$\tan(r) = \frac{x}{A} \cong \sin(r) \text{ and } \tan(i) = \frac{x}{R} \cong \sin(i)$$

$$\therefore n = \frac{\sin(r)}{\sin(i)} \cong \frac{\left(\frac{x}{A}\right)}{\left(\frac{x}{R}\right)} = \frac{R}{A} = \frac{\text{Real depth}}{\text{Apparent depth}}$$

2) A stick or pencil kept obliquely in a glass containing water appears broken as its part in water appears to be raised.

> Small angle approximation: For small angles, expressed in radian, sin $\theta \cong \theta \cong \tan\theta$.
> For example, for
>
> $$\theta = 30^0 = \left(\frac{\pi}{6}\right)^c = 0.5236^c,$$
>
> we have $\sin\theta = 0.5$
>
> In this case the error is $0.5236 - 0.5 = 0.0236$ in 0.5, which is 4.72 %.
> For practical purposes we consider angles less than 10^0 where the error in using sin $\theta \cong \theta$ is less than 0.51 %. (Even for 60^0, it is still 15.7 %)
> It is left to you to verify that this is almost equally valid for $\tan\theta$ till 20^0 only.

Example 9. 4: A crane flying 6 m above a still, clear water lake sees a fish underwater. For the crane, the fish appears to be 6 cm below the water surface. How much deep should the crane immerse its beak to pick that fish?

For the fish, how much above the water surface does the crane appear? Refractive index of water = 4/3.

Solution: For crane, apparent depth of the fish is 6 cm and real depth is to be determined.

For fish, real depth (height, in this case) of the crane is 6 m and apparent depth (height) is to be determined.

$$n = \frac{R}{A} = \frac{\text{Real depth}}{\text{Apparent depth}}$$

For crane, it is water with respect to air as real depth is in water and apparent depth is as seen from air

$$\therefore n = \frac{4}{3} = \frac{R}{A} = \frac{R}{6} \quad \therefore R = 8\,\text{cm}$$

For fish, it is air with respect to water as the real height is in air and seen from water.

$$\therefore n = \frac{3}{4} = \frac{R}{A} = \frac{6}{A} \quad \therefore A = 8\,\text{m}$$

9.6 Total internal reflection:

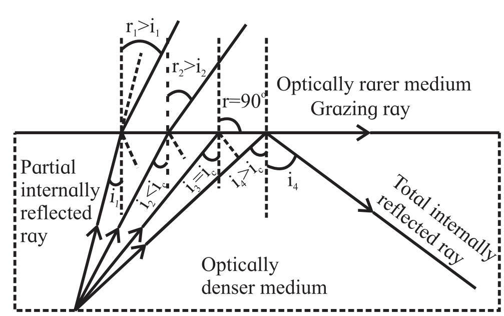


Fig. 9.7: Total internal reflection.

Figure 9.7 shows refraction of light emerging from a denser medium into a rarer medium for various angles of incidence. The angles of refraction in the rarer medium are larger than the corresponding angles of incidence. At a particular angle of incidence i_c in the denser medium, the corresponding angle of refraction in the rarer medium is 90^0. For angles of incidence greater than i_c, the angle of refraction become larger than 90^0 and the ray does not enter into rarer medium at all but is reflected totally into the denser medium. This is called total internal reflection. In general, there is always partial reflection and partial refraction at the interface. During total internal reflection TIR, it is total reflection and no refraction. The corresponding angle of incidence in the denser medium is greater than or equal to the critical angle.

Critical angle for a pair of refracting media can be defined as that angle of incidence in the denser medium for which the angle of refraction in the rarer medium is 90°.

Do you know ?

In Physics the word *critical* is used when certain phenomena are not applicable or more than one phenomenon are applicable. Some examples are as follows.

(i) In case of total internal reflection, the phenomenon of reversibility of light is not applicable at critical angle and refraction is possible only for angles of incidence in the denser medium smaller than the critical angle.

(ii) At the *critical* temperature, a substance coexists into all the three states; solid, liquid and gas. At all the other temperatures, only two states are simultaneously possible.

(iii) For liquids, streamline flow is possible till *critical* velocity is achieved. At critical velocity it can be either streamline or turbulent.

Let μ be the relative refractive index of denser medium with respect to the rarer. Applying Snell's law at the critical angle of incidence, i_C, we can write $\sin(i_c) = \frac{1}{\mu}$ as,

$$(\mu)\sin(i_c) = (1)\sin 90°$$

For commonly used glasses of

$\mu = 1.5$, $i_c = 41° \, 49' \cong 42°$ and for water of

$\mu = \frac{4}{3}$, $i_c = 48° \, 35'$ (Both, with respect to air)

9.6.1 Applications of total internal reflection:

(i) Optical fibre: Though little costly for initial set up, optic fibre communication is undoubtedly the most effective way of telecommunication by way of EM waves.

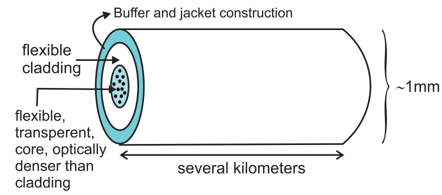


Fig. 9.8 (a): Optical fibre construction.

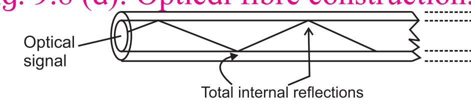


Fig. 9.8 (b): Optical fibre working.

An optical fibre essentially consists of an extremely thin (slightly thicker than a human hair), transparent, flexible core surrounded by optically rarer (smaller refractive index), flexible cover called cladding. This system is coated by a buffer and a jacket for protection. Entire thickness of the fibre is less than half a mm. (Fig. 9.8(a)). Number of such fibres may be packed together in an outer cover.

An optical signal (ray) entering the core suffers multiple total internal reflections (Fig. 9.8 (b)) and emerges after several kilometers with extremely low loss travelling with highest possible speed in that material (~2,00,000 km/s for glass). Some of the advantages of optic fibre communication are listed below.

(a) Broad bandwidth (frequency range): For TV signals, a single optical fibre can carry over 90000 channels (independent signals).

(b) Immune to EM interference: Being electrically non-conductive, it is not able to pick up nearby EM signals.

(c) Low attenuation loss: The loss is lower than 0.2 dB/km so that a single long cable can be used for several kilometers.

(d) Electrical insulator: No issue with ground loops of metal wires or lightning.

(e) Theft prevention: It is does not use copper or other expensive material.

(f) Security of information: Internal damage is most unlikely.

(ii) Prism binoculars: Binoculars using only two cylinders have a limitation of field of view as the distance between the two cylinders can't be greater than that between the two eyes. This limitation can be overcome by using two right angled glass prisms ($i_C \sim 42^0$) used for total internal reflection as shown in the Fig. 9.9. Total internal reflections occur inside isosceles, right angled prisms.

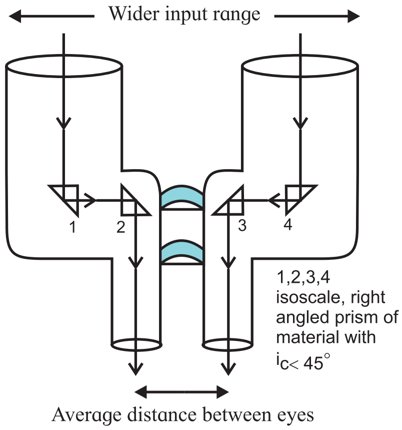


Fig. 9.9: Prism binoculars

(iii) Periscope: It is used to see the objects on the surface of a water body from inside water. The rays of light should be reflected twice through right angle. Reflections are similar to those in the binoculars (Fig 9.10) and total internal reflections occur inside isosceles, right angled prisms.

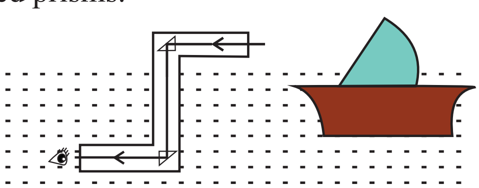
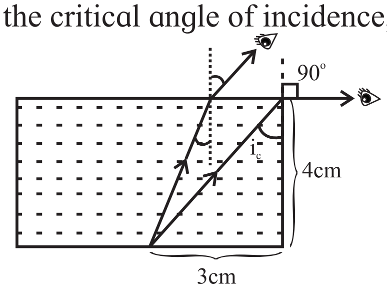


Fig. 9.10: Periscope.

Example 9.5: There is a tiny LED bulb at the center of the bottom of a cylindrical vessel of diameter 6 cm. Height of the vessel is 4 cm. The beaker is filled completely with an optically dense liquid. The bulb is visible from any inclined position but just visible if seen along the edge of the beaker. Determine refractive index of the liquid.

Solution: As seen from the accompanying figure, if the bulb is just visible from the edge, angle of incidence in the liquid (at the edge) must be the critical angle of incidence, i_C



From the dimensions given,

$$\tan(i_c) = \frac{3}{4} \therefore \sin(i_c) = \frac{3}{5} \therefore n_{\text{liquid}} = \frac{\sin 90^\circ}{\sin(i_c)} = \frac{5}{3}$$

9.7 Refraction at a spherical surface and lenses:

In the section 9.5 we saw that due to refraction, the bottom of a water body appears to be raised and $n_{water} = \dfrac{Real\ depth}{apparent\ depth}$.

However, this is valid only if we are dealing with refraction at a plane surface. In many cases such as liquid drops, lenses, ellipsoid paper weights, etc, curved surfaces are present and the formula mentioned above may not be true. In such cases we need to consider refraction at one or more spherical surfaces. This will involve parameters including the curvature such as radius of curvature, in addition to refractive indices.

Lenses: Commonly used lenses can be visualized to be consisting of intersection of two spheres of radii of curvature R_1 and R_2 or of one sphere and a plane surface ($R = \infty$) . A lens is said to convex if it is thicker in the middle and narrowing towards the periphery. A lens is concave if it is thicker at periphery and narrows down towards center. Convex lens is visualized to be internal cross section of two spheres (or one sphere and a plane surface) while concave lens is their external cross section (Figs. 9.11-a to 9.11-f). Concavo-convex and convexo-concave lenses are commonly used for spectacles of positive and negative numbers, respectively.

For lenses of material optically denser than the medium in which those are kept, convex lenses have positive focal length [according to Cartesian sign convention] and converge the incident beam while concave lenses have negative focal length and diverge the incident beam.

For most of the applications of lenses, maximum thickness of lens is negligible (at least 50 times smaller) compared with all the other distances such as R_1 and R_2, u, v, f, etc. Such a lens is called as a thin lens and physical center of such a lens can be assumed to be the common pole (or optical center) for both its refracting surfaces.

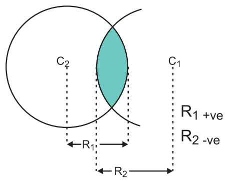


Fig. 9.11 (a): Convex lens as internal cross section of two spheres.

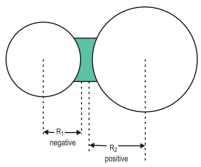


Fig. 9.11 (b): Concave lens as external cross section of two spheres.

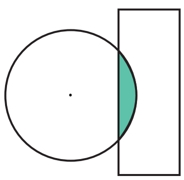


Fig. 9.11 (c): Plano convex lens

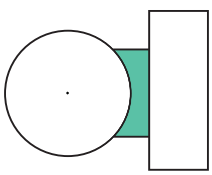


Fig. 9.11 (d): Plano concave lens

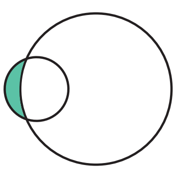


Fig. 9.11 (e): concave-convex lens

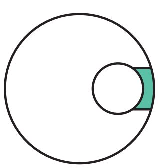


Fig. 9.11 (f) convex-concave lens

For any thin lens, $\frac{1}{f}=\frac{1}{\mathrm{v}}-\frac{1}{u}$ --- (9.2)

If necessary, we can have a number of thin lenses in contact with each other having common principal axis. Focal power of such combination is given by the algebraic addition (by considering ± signs) of individual focal powers.

$$\therefore \frac{1}{f}=\sum\left(\frac{1}{f_i}\right)=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\ldots$$

$$=P_1+P_2+P_3\ldots..=\sum P_i=P$$

For only two thin lenses, separated in air by distance d,

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}=P_1+P_2-dP_1P_2=P$$

For lenses, the relations between u, v, R and f depend also upon the refractive index n of the material of the lens. The relation $\left(f=\frac{R}{2}\right)$ does NOT hold good for lenses. Below we shall derive the necessary relation by considering refraction at the two surfaces of a lens independently.

Unless mentioned specifically, we assume lenses to be made up of optically denser material compared to the medium in which those are kept, e.g., glass lenses in air or in water, etc. As special cases we may consider lenses of rarer medium such as an air lens in water or inside a glass. A spherical hole inside a glass slab is also a lens of rarer medium. In such case, physically (or geometrically or shape-wise) convex lens diverges the incident beam while concave lens converges the incident beam.

Refraction at a single spherical surface: Consider a spherical surface YPY' of radius of curvature R, separating two transparent media of refractive indices n_1 and n_2 respectively with $n_1 < n_2$. P is the pole and X'PX is the principal axis. A point object O is at an object distance -u from the pole, in the medium of refractive index n_1. Convexity or concavity of a surface is always with respect to the incident rays, i.e., with respect to a real object. Hence in this case the surface is convex (Fig. 9.12).

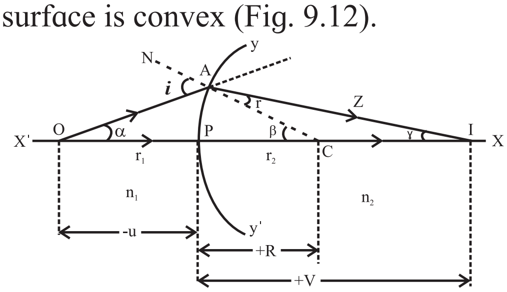


Fig. 9.12: Refraction at a single refracting surface.

To locate its image and in order to minimize spherical aberration, we consider two paraxial rays. The ray OP along the principal axis travels undeviated along PX. Another ray OA strikes the surface at A. CAN is the normal from center of curvature C of the surface at A. Angle of incidence in the medium n_1 at A is i.

As $n_1 < n_2$, the ray deviates towards the normal, travels along AZ and cuts the principal axis at I. Thus, real image of point object O is formed at I. Angle of refraction in medium n_2 is r. According to Snell's law,

$$n_1 \sin(i) = n_2 \sin(r) \qquad \text{--- (9.3)}$$

Let α, β and γ be the angles subtended by incident ray, normal and refracted ray with the principal axis.

$\therefore i = \alpha + \beta$ and $r = \beta - \gamma$

For paraxial rays, all these angles are small and PA can be considered as an arc for α, β and γ.

$\therefore \sin(i) \cong i$ and $\sin(r) \cong r$

Also, $\alpha \cong \dfrac{arc\ AP}{PO} = \dfrac{arc\ AP}{-u}$,

$\beta = \dfrac{arc\ AP}{PC} = \dfrac{arc\ AP}{R}$ and

$\gamma \cong \dfrac{arc\ AP}{PI} = \dfrac{arc\ AP}{\text{v}}$

$\therefore n_1 i = n_2 r$

$\therefore n_1(\alpha + \beta) = n_2(\beta - \gamma)$

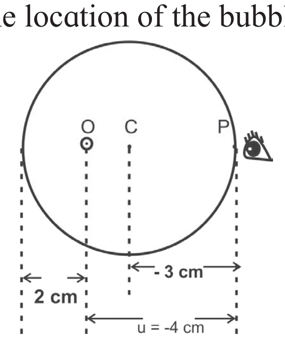
$\therefore (n_2 - n_1)\beta = n_2\gamma + n_1\alpha$

Substituting α, β and γ and canceling 'arc AP', we get

$$\frac{n_2 - n_1}{R} = \frac{n_2}{\text{v}} - \frac{n_1}{u} \qquad \text{--- (9.4)}$$

Example 6: A glass paper-weight (n =1.5) of radius 3 cm has a tiny air bubble trapped inside it. Closest distance of the bubble from the surface is 2 cm. Where will it appear when seen from the other end (from where it is farthest)?

Solution: Accompanying Figure below illustrates the location of the bubble.



According to the symbols used in the Eq. (9.4), we get,

n_1 = refractive index of the medium of real object(medium of incident rays) = 1.5

n_2 = refractive index of the other medium = 1

$u = -4\,cm$

$\text{v} = ?$

$R = -3\,cm$

$$\frac{n_2 - n_1}{R} = \frac{n_2}{\text{v}} - \frac{n_1}{u}$$

$$\therefore \frac{1-1.5}{-3} = \frac{1}{\text{v}} - \frac{1.5}{-4} \quad \therefore \frac{1}{6} = \frac{1}{\text{v}} + \frac{3}{8} \quad \therefore \text{v} = -4.8\,\text{cm}$$

In this case apparent depth is NOT less than real depth. This is due to curvature of the refracting surface.

In this case (Fig. 9.12) we had considered the object placed in rarer medium, real image in denser medium and the surface facing the object to be convex. However, while deriving the relation, all the symbolic values (which could be numeric also) were substituted as per the Cartesian sign convention (e.g. 'u' as negative, etc.). Hence the final expression (Eq. 9.4) is applicable to any surface separating any two media, and real or virtual image provided you substitute your values (symbolic or numerical) as per Cartesian sign convention. The *only restriction* is that n_1 is for medium of real object and n_2 is the *other* medium (*not necessarily* the medium of image). Only in the case of real image, it will be in medium n_2. If virtual, it will be in the medium n_1 (with image distance negative how do you justify this?).

We strongly suggest you to do the derivations yourself for any other special case such as object placed in the denser medium, virtual image, concave surface, etc. It must be remembered that in any case you will land up with the same expression as in Eq. (9.4).

Lens makers' equation: Relation between refractive index (n), focal length (f) and radii of curvature R_1 and R_2 for a thin lens.

Consider a lens of radii of curvature R_1 and R_2 kept in a medium such that n is refractive index of material of the lens with respect to the outside medium. Assuming the lens to be thin, P is the common pole for both the surfaces. O is a point object on the principal axis at a distance u from P. First refracting surface of the lens of radius of curvature R_1 faces the object (Fig 9.13).

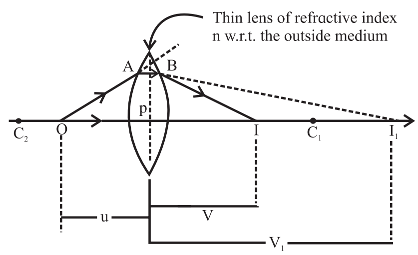


Fig. 9.13: Lens maker's equation.

Axial ray OP travels undeviated. Paraxial ray OA deviates towards normal and would intersect axis at I_1, in the absence of second refracting surface. $PI_1 = v_1$ is the image distance for intermediate image I_1.

Thus, the symbols to be used in Eq. (9.4) are

$$n_2 = n, \quad n_1 = 1, \quad R = R_1, \quad u = u, \quad v = v_1$$

$$\therefore \frac{n-1}{R_1} = \frac{n}{v_1} - \frac{1}{(u)} \qquad \text{--- (9.5)}$$

(Not that, in this case, we are not substituting the algebraic values but just using different symbols.)

Before reaching I_1, the ray PI_1 is intercepted at B by the second refracting surface. In this case, the incident rays AB and OP are in the medium of refractive index n and converging towards I_1. Thus, I_1 acts as *virtual* object for second surface of radius of curvature (R_2) and object distance is $(u = v_1)$. As the incident rays are in the medium of refractive index n, this is the medium of (virtual) object $\therefore n_1 = n$ and refractive index of the other medium is $n_2 = 1$.

After refraction, the ray bends away from the normal and intersects the principal axis at I which is the real image of object O formed due to the lens. $\therefore$ PI = v.

Substituting all these *symbols* in Eq. (9.4), we get

$$\frac{1-n}{R_2} = \frac{n-1}{-R_2} = \frac{1}{v} - \frac{n}{(v_1)} \qquad \text{--- (9.6)}$$

Adding Eq. (9.5) and (9.6), we get,

$$(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{v} - \frac{1}{u}$$

For $u = \infty, v = f$

$$\therefore \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \text{--- (9.7)}$$

For preparing spectacles, it is necessary to grind the glass (or acrylic, etc.) for having the desired radii of curvature. Equation (9.7) can be used to calculate the radii of curvature for the lens, hence it is called the lens makers' equation. (It should be remembered that while solving problems when you are using equations 9.1, 9.2, 9.4, 9.7, etc., we will be substituting the values of the corresponding quantities. Hence this time it is algebraic substitution, i.e., with proper $\pm$ sign)

Special cases:

Most popular and most common special case is the one in which we have a thin, symmetric, double lens. In this case, R_1 and R_2 are numerically equal.

(A) Thin, symmetric, double convex lens: R_1 is positive, R_2 is negative and numerically equal. Let $|R_1| = |R_2| = R$.

$$\therefore \frac{1}{f} = (n-1)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{2(n-1)}{R}$$

Further, for popular variety of glasses, $n \cong 1.5$. In such a case, $f = R$.

(B) Thin, symmetric, double concave lens: R_1 is negative, R_2 is positive and numerically equal. Let $|R_1| = |R_2| = R$.

$$\therefore \frac{1}{f} = (n-1)\left(\frac{1}{-R} - \frac{1}{R}\right) = \frac{2(n-1)}{-R}$$

Further if $n \cong 1.5, f = -R$

(C) Thin, planoconvex lenses: One radius is R and the other is ∞. $\therefore \frac{1}{f} = \frac{n-1}{R}$

Further if $n \cong 1.5, f = 2|R|$

Example 7: A dense glass double convex lens $(n = 2)$ designed to reduce spherical aberration has $|R_1|:|R_2|=1:5$. If a point object is kept 15 cm in front of this lens, it produces its real image at

7.5 cm. Determine R_1 and R_2.

Solution: u = - 15 cm, v = + 7.5 cm (real image is on opposite side).

$$\frac{1}{f}=\frac{1}{\mathrm{v}}-\frac{1}{u} \quad \therefore \frac{1}{f}=\frac{1}{7.5}-\frac{1}{-15} \quad \therefore f=+5\,\mathrm{cm}$$

The lens is double convex. Hence, R_1 is positive and R_2 is negative. Also, $|R_2|=5|R_1|$ and $n=2$.

$$(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\frac{1}{f}$$

$$\therefore (2-1)\left(\frac{1}{R_1}-\frac{1}{-(5R_1)}\right)=\frac{1}{5}$$

$$\therefore (1)\left(\frac{6}{5R_1}\right)=\frac{1}{5} \quad \therefore R_1=6\,\mathrm{cm} \quad \therefore R_2=30\,\mathrm{cm}$$

9.8 Dispersion of light and prisms:

The colour of light that we see depends upon the frequency of that ray (wave). The refractive index of a material also depends upon the frequency of the wave and increases with frequency. Obviously refractive index of light is different for different colours. As a result, for an obliquely incident ray, the angles of refraction are different for each colour and they separate (disperse) as they travel along different directions. This phenomenon is called angular dispersion Fig 9.14.

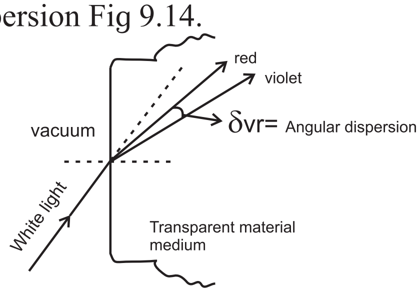


Fig. 9.14: Angular dispersion at a single surface.

If a polychromatic beam of light (bundle of rays of different colours) is obliquely incident upon a plane parallel transparent slab, emergent beam consists of all component colours separated out. However, in this case all those are parallel to each other and also parallel to initial direction. This is lateral dispersion which is measured as the perpendicular distance between the direction of incident ray and respective directions of dispersed emergent rays (L_R and L_V) Fig 9.15. For it to be easily detectable, the parallel surfaces must be separated over very large distance and i should be large.

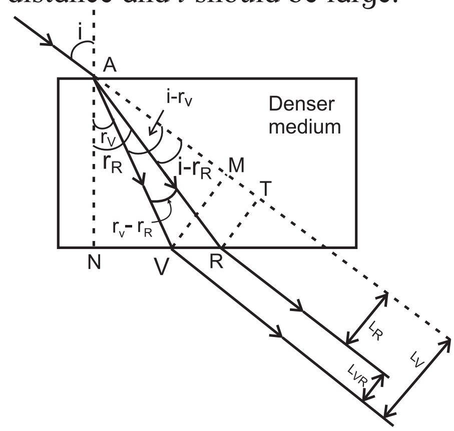


Fig. 9.15: Lateral dispersion due to plane parellal slab.

Example 8: A fine beam of white light is incident upon the longer side of a plane parallel glass slab of breadth 5 cm at angle of incidence 60^0. Calculate angular deviation of red and violet rays within the slab and lateral dispersion between them as they emerge from the opposite side. Refractive indices of the glass for red and violet are 1.51 and 1.53 respectively.

Solution: As shown in the Fig. 9.15 above, VM = L_V and RT = L_R give respective lateral deviations for violet and red colours and L_{VR} = $L_V - L_R$ is the lateral dispersion between these colours. n_R = 1.51, n_V = 1.53 and i = 60°

$$\therefore \sin r_R=\frac{\sin i}{n_R}=\frac{\sin 60^0}{1.51}=0.5735$$

$$\sin r_V=\frac{\sin i}{n_V}=\frac{\sin 60^0}{1.53}=0.566$$

$$\therefore r_R=35^0 \text{ and } R_V=34^0 28' \quad \therefore \delta_{RV}=r_R-r_V\, 32'$$

$$\therefore i-r_R=25^0,\; i-r_V=25^0 32'$$

$$\therefore \mathrm{AR}=\frac{\mathrm{AN}}{\cos r_R}=6.104\,\mathrm{cm}$$

$$\mathrm{AV}=\frac{\mathrm{AN}}{\cos r_V}=6.063\,\mathrm{cm}$$

$$\therefore L_R=\mathrm{RT}=\mathrm{AR}\left(\sin\left[i-r_R\right]\right)=2.58\,\mathrm{cm}$$

$$L_V=\mathrm{VM}=\mathrm{AV}\left(\sin\left[i-r_V\right]\right)=2.58\,\mathrm{cm}$$

$$\therefore L_{VR}=L_V-L_R=0.033\,\mathrm{cm}=0.33\text{ mm}$$

It shows that the lateral dispersion is too small to detect.

In order to have appreciable and observable dispersion, two parallel surfaces are not useful. In such case we use prisms, in which two refracting surfaces inclined at an angle are used. Popular variety of prisms are having three rectangular surfaces forming a triangle. At a time two of these are taking part in the refraction. The one, not involved in refraction is called base of the prism. Fig 9.16.

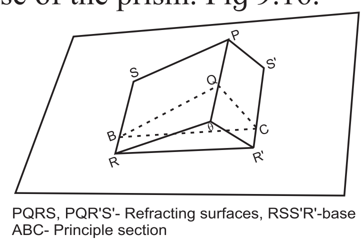


Fig. 9.16: Prism consisting of three plane surfaces.

Any section of prism perpendicular to the base is called principal section of the prism. Usually we consider all the rays in this plane. Fig 9.17 a and 9.17 b show refraction through a prism for monochromatic and white beams respectively. Angular dispersion is shown for white beam.

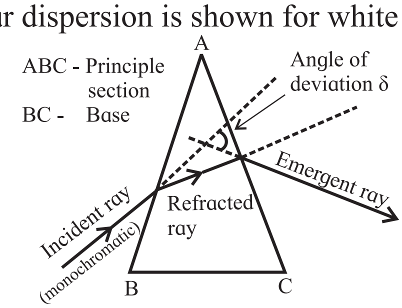


Fig. 9.17 (a): Refraction through a prism (monocromatic light).

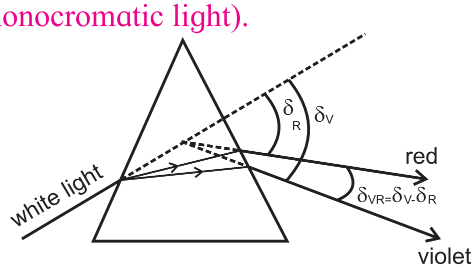


Fig. 9.17 (b): Angular dispersion through a prism. (white light).

Relations between the angles involved: Figure 9.18 shows principal section ABC of a prism of absolute refractive index n kept in air. Refracting surfaces AB and AC are inclined at angle A, which is refracting angle of prism or simply 'angle of prism'. Surface BC is the base. A monochromatic ray PQ obliquely strikes first reflecting surface AB. Normal passing through the point of incidence Q is MQN. Angle of incidence at Q is i. After refraction at Q, the ray deviates towards the normal and strikes second refracting surface AC at R which is the point of emergence. MRN is the normal through R. Angles of refraction at Q and R are r_1 and r_2 respectively.

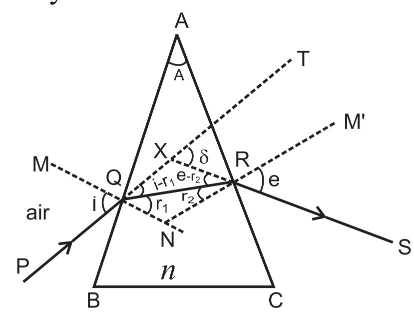


Fig. 9.18: Deviation through a prism.

After R, the ray deviates away from normal and finally emerges along RS making e as the angle of emergence. Incident ray PQ is extended as QT. Emergent ray RS meets QT at X if traced backward. Angle TXS is angle of deviation δ .

$\angle$ AQN = $\angle$ ARN = 90^0 ...... (Angles at normal)

$\therefore$ From quadrilateral AQNR,

$$A + \angle QNR = 180^0 \quad \text{--- (9.8)}$$

From Δ QNR, $r_1 + r_2 + \angle QNR = 180^0$ --- (9.9)

$\therefore$ From Eqs. (9.8) and (9.9),

$$A = r_1 + r_2 \quad \text{--- (9.10)}$$

Angle δ is exterior angle for triangle XQR.

$$\therefore \angle XQR + \angle XRQ = \delta$$

$$\therefore (i - r_1) + (e - r_2) = \delta$$

$$\therefore (i + e) - (r_1 + r_2) = \delta$$

Hence, using Eq. (9.10), $(i+e)-(A)=\delta$

$$\therefore i + e = A + \delta \quad \text{--- (9.11)}$$

Deviation curve, minimum deviation and prism formula: From the relations (9.10) and (9.11), it is clear that δ, e, r_1 and r_2 depend upon i, A and n. After a certain minimum value of angle of incidence i_{min}, the emergent ray is possible. This is because of the fact that for $i < i_{min}$, $r_2 > i_c$ and there is total internal reflection at the second surface and there is no emergent ray. This will be shown later. Then onwards,